Chapter 6 Magnetic Fields in Matter

6.1 Magnetization

6.1.1 Diamagnets, Paramagnets, Ferromagnets

All the magnetic phenomena are due to electric charges in motion:

Electrons orbiting around nuclei
Electrons spinning about their axes } magnetic dipoles

When a magnetic field is applied, a net alignment of these magnetic dipoles occurs, and the medium becomes magnetically polarized, or magnetized.

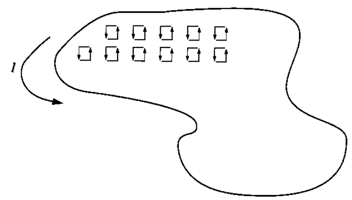
The magnetic polarization $\mathbf{M}$, unlike electrical polarization $\mathbf{P}$, might be parallel to $\mathbf{B}$ (*paramagnets*) or opposite to $\mathbf{B}$ (*diamagnets*).

A few substances (*ferromagnets*) retain their magnetization even after the external field has been removed.

6.1.2. Torques and Forces on Magnetic Dipoles

A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole does in an electric field.

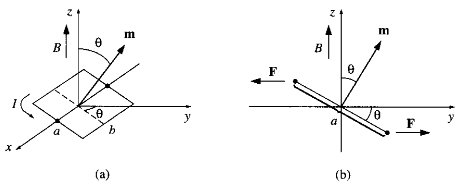
Any current loop could be built up from infinitesimal *rectangles*, with all the "internal" side canceling. There is no actual loss of generality in using the shape.



Let's calculate the torque on a rectangular current loop in a uniform magnetic field.

Torques and Forces on Magnetic Dipoles

Center the loop at the origin, and tilt it an angle θ from the z axis towards the y axis. Let $\mathbf{B}$ point in the z direction.



Sloping sides: the forces cancel.
Horizontal sides: the forces cancel but they generate a torque.

$$\mathbf{N} = \mathbf{L} \times \mathbf{F} = aF \sin\theta \hat{\mathbf{x}}$$

The magnitude of the force on each of these segments is:

$$F = IbB$$

Torques and Dipole Moment

$$\mathbf{N} = IabB \sin\theta \hat{\mathbf{x}} = mB \sin\theta \hat{\mathbf{x}} = \mathbf{m} \times \mathbf{B}$$

where $m = Iab$ is the magnetic dipole moment of the loop.

This equation is identical in form to the electrical analogy.

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

The torque is again in such a direction as to line the dipole up parallel to the field (***paramagnetism***).

QM: The Pauli exclusion principle dictates that the electrons within a given atom lock together in pairs with opposite spins, and this effectively neutralizes the torque on the combination.

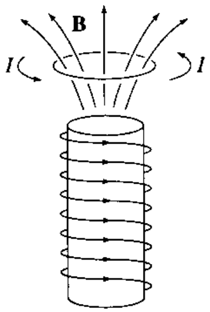
Forces in Nonuniform Magnetic Field

In a uniform field, the net force on a current loop is zero:

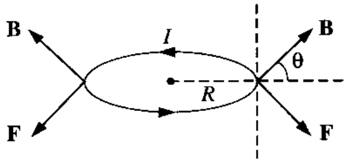
$$F = I\oint (d\mathbf{l} \times \mathbf{B}) = I\oint (d\mathbf{l}) \times \mathbf{B} = 0$$

=0

In a nonuniform field this is no longer the case, because the magnetic field B could not come outside the integral.



Fringing field effect: $F = 2\pi IRB\cos\theta$

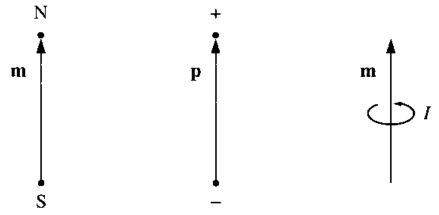


Forces on an Infinitesimal Current Loop and Model

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) \qquad (= -\nabla U\text{, where } U = -(\mathbf{m}\cdot\mathbf{B}))$$

Identical to the electrical formula $\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{E})$

Does the magnetic dipole consist of a pair of opposite magnetic monopoles just like an electric dipole?



(a) Magnetic dipole (Gilbert model) (b) Electric dipole (c) Magnetic dipole (Ampère model)

6.1.3. Effect of a Magnetic Field on Atomic Orbits

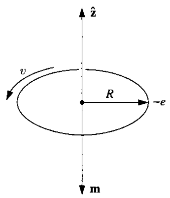
Electrons not only spin; they also revolve around the nucleus.

Let's assume the orbit is a circle of radius R. The current looks like steady (really?)

Current $I = \dfrac{e}{T} = \dfrac{e}{\frac{2\pi R}{v}} = \dfrac{ev}{2\pi R}$

The negative charge of the electron

Orbital dipole moment $\mathbf{m} = -\dfrac{evR}{2}\hat{\mathbf{z}} \quad (m = I\pi R^2)$

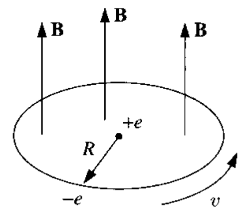


Electron Speeds Up or Slows Down

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} = m_e\frac{v^2}{R}$$ without the magnetic field.

The centripetal force comes from two sources: *the electric force* and *the magnetic force*.

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R'^2} + ev'B = m_e\frac{v'^2}{R'}$$



Assume $R' \cong R$

$$ev'B = \frac{m_e(v'^2 - v^2)}{R} = \frac{m_e(v'+v)(v'-v)}{R}$$

$$\therefore \Delta v = (v' - v) \approx \frac{eRB}{2m_e}$$ When **B** is turn on, the electron speeds up.

The Dipole Moment and The Diamagnetism

A change in the orbital speed means a change in the dipole moment

$$\Delta \mathbf{m} = -\frac{1}{2} e \Delta v R = -\frac{e^2 R^2}{2 m_e} \mathbf{B}$$

The change in $\mathbf{m}$ is opposite to the direction of $\mathbf{B}$.

In the presence of a magnetic field, each atom picks up a little "extra" dipole moment, and the increments are all antiparallel to the field. This is the mechanism responsible for **diamagnetism**.

This is a universal phenomenon, affecting all atoms, but it is typically much *weaker* than **paramagnetism**.

6.1.4 Magnetization

In the present of a magnetic field, matter becomes magnetized. Upon microscopic examination, it contains many tiny dipoles, with a net alignment along some direction.

Two mechanisms account for this magnetic polarization:

1.Paramagnetism: the dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to the field.

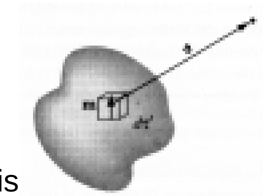
2. Diamagnetism: the orbital speed of the electrons is altered in such a way as to change the orbital dipole moment in the direction opposite to the field.

We describe the state of magnetic polarization by the vector quantity:

$$\mathbf{M} \equiv \text{magnetic dipole moment per unit volume.}$$

6.2 The Field of a Magnetized Object
6.2.1 Bound Currents

Suppose we have a piece of magnetized material (i.e. $\mathbf{M}$ is given). What field does this object produce?



The vector potential of a single dipole $\mathbf{m}$ is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

In the magnetized object, each volume element carries a dipole moment $\mathbf{M} d\tau'$, so the total vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

Vector potential and Bound Currents

Can the equation be expressed in a more illuminating form, as in the electrical case? Yes!

By exploiting the identity,

$$\nabla' \frac{1}{\mathscr{r}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$(\hat{\mathbf{x}}' \frac{\partial}{\partial x'} + \hat{\mathbf{y}}' \frac{\partial}{\partial y'} + \hat{\mathbf{z}}' \frac{\partial}{\partial z'}) \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \frac{\hat{\mathbf{x}}'(x-x') + \hat{\mathbf{y}}'(y-y') + \hat{\mathbf{z}}'(z-z')}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{3/2}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

The vector potential is $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{M}(\mathbf{r}') \times (\nabla' \frac{1}{\mathscr{r}}) d\tau'$

Using the product rule $\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f(\nabla \times \mathbf{A})$

and integrating by part, we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left\{ \int \frac{1}{\mathscr{r}} [\nabla' \times \mathbf{M}(\mathbf{r}')] d\tau' - \int \nabla' \times [\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}] d\tau' \right\}$$

↓ how? Prob. 1.60

$$= \frac{\mu_0}{4\pi} \left\{ \int \frac{1}{\mathscr{r}} [\nabla' \times \mathbf{M}(\mathbf{r}')] d\tau' \right\} + \frac{\mu_0}{4\pi} \oint \frac{1}{\mathscr{r}} [\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}'] da'$$

Problem 1.60 Although the gradient, divergence, and curl theorems are the fundamental integral theorems of vector calculus, it is possible to derive a number of corollaries from them. Show that:

(a) $\int_{\mathcal{V}}(\nabla T)\,d\tau = \oint_{\mathcal{S}} T\,d\mathbf{a}$. [*Hint:* Let $\mathbf{v} = \mathbf{c}T$, where $\mathbf{c}$ is a constant, in the divergence theorem; use the product rules.]

(b) $\int_{\mathcal{V}}(\nabla \times \mathbf{v})\,d\tau = -\oint_{\mathcal{S}} \mathbf{v} \times d\mathbf{a}$. [*Hint:* Replace $\mathbf{v}$ by $(\mathbf{v} \times \mathbf{c})$ in the divergence theorem.]

(c) $\int_{\mathcal{V}}[T\nabla^2 U + (\nabla T)\cdot(\nabla U)]\,d\tau = \oint_{\mathcal{S}}(T\nabla U)\cdot d\mathbf{a}$. [*Hint:* Let $\mathbf{v} = T\nabla U$ in the divergence theorem.]

Gauss's law $\int_v (\nabla\cdot\mathbf{E})d\tau = \oint_S \mathbf{E}\cdot d\mathbf{a}$

Let $\mathbf{E} = \mathbf{v}\times\mathbf{c}$, $\begin{cases} \int_v (\nabla\cdot(\mathbf{v}\times\mathbf{c}))d\tau = \mathbf{c}\cdot\int_v (\nabla\times\mathbf{v})d\tau \\ \oint_S (\mathbf{v}\times\mathbf{c})\cdot d\mathbf{a} = -\mathbf{c}\cdot\oint_S \mathbf{v}\times d\mathbf{a} \end{cases}$

Since $\mathbf{c}$ is a constant vector, so $\int_v (\nabla\times\mathbf{v})d\tau = -\oint_S \mathbf{v}\times d\mathbf{a}$

Vector potential and Bound Currents

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{1}{\text{r}}[\nabla'\times\mathbf{M}(\mathbf{r}')]d\tau' + \frac{\mu_0}{4\pi}\oint\frac{1}{\text{r}}[\mathbf{M}(\mathbf{r}')\times\hat{\mathbf{n}}']da'$$

$\mathbf{J}_b = \nabla'\times\mathbf{M}(\mathbf{r}')$ volume current

$\mathbf{K}_b = \mathbf{M}(\mathbf{r}')\times\hat{\mathbf{n}}'$ surface current

bound currents

With these definitions,

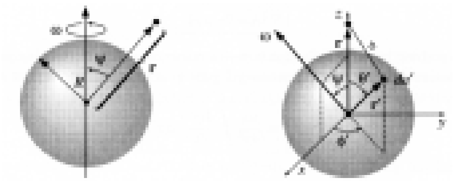
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_v \frac{\mathbf{J}_b}{\text{r}}d\tau' + \frac{\mu_0}{4\pi}\oint_S \frac{\mathbf{K}_b}{\text{r}}da'$$

The electrical analogy

volume charge density $\rho_b = -\nabla\cdot\mathbf{P}$

surface charge density $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$

Example 5.11 A spherical shell, of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω. Find the vector potential it produce at point $\mathbf{r}$.



Sol: First, let the observer is in the z axis and ω is tilted at an angle ψ

Vector potential is $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{K}(\mathbf{r}')}{\text{r}}da'$

The surface current density $\mathbf{K}(\mathbf{r}') = \sigma\mathbf{v}'$

$$\mathbf{v}' = \boldsymbol{\omega}\times\mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega[-(\cos\psi\sin\theta'\sin\phi')\,\hat{\mathbf{x}} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\,\hat{\mathbf{y}} + (\sin\psi\sin\theta'\sin\phi')\,\hat{\mathbf{z}}].$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{R\omega(-\cos\psi\sin\theta'\sin\phi')\hat{\mathbf{x}}}{\sqrt{r^2+R^2-2rR\cos\theta'}}R^2\sin\theta' d\theta' d\phi'$$
$$+\frac{\mu_0}{4\pi}\int\frac{R\omega(\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\hat{\mathbf{y}}}{\sqrt{r^2+R^2-2rR\cos\theta'}}R^2\sin\theta' d\theta' d\phi'$$
$$+\frac{\mu_0}{4\pi}\int\frac{R\omega(\sin\psi\sin\theta'\sin\phi')\hat{\mathbf{z}}}{\sqrt{r^2+R^2-2rR\cos\theta'}}R^2\sin\theta' d\theta' d\phi'$$

$$\mathbf{A}(\mathbf{r}) = \frac{-R^3\sigma\omega\sin\psi\mu_0\hat{\mathbf{y}}}{4\pi}\int\frac{\cos\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}}\sin\theta' d\theta' d\phi'$$
$$= \frac{-R^3\sigma\omega\sin\psi\mu_0\hat{\mathbf{y}}}{4\pi}(2\pi)\int_0^{\pi}\frac{-\cos\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}}d\cos\theta'$$
$$= \frac{-\mu_0 R^3\sigma\omega\sin\psi\hat{\mathbf{y}}}{2}\int_{-1}^{1}\frac{u}{\sqrt{r^2+R^2-2rRu}}du$$

$$\int_{-1}^{+1}\frac{u}{\sqrt{R^2+r^2-2Rru}}du = -\frac{(R^2+r^2+Rru)}{3R^2r^2}\sqrt{R^2+r^2-2Rru}\,\Big|_{-1}^{+1}$$
$$= -\frac{1}{3R^2r^2}\Big[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\Big].$$

$$\mathbf{A}(\mathbf{r}) = \frac{-\mu_0 R^3 \sigma\omega \sin\psi \hat{\mathbf{y}}}{2}\left(-\frac{(R^2+r^2+Rr)|R-r|-(R^2+r^2-Rr)(R+r)}{3R^2r^2}\right)$$

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{2}(\boldsymbol{\omega}\times\mathbf{r}) & \text{inside} \\ \dfrac{\mu_0 R^4\sigma}{2r^3}(\boldsymbol{\omega}\times\mathbf{r}) & \text{outside} \end{cases}$$

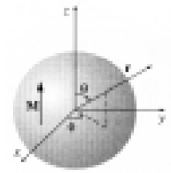
Reverting to the “natural” coordinate, we have

$$\mathbf{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r \sin\theta\, \hat{\boldsymbol{\phi}}, & (r \le R), \\ \dfrac{\mu_0 R^4 \omega\sigma}{3} \dfrac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}}, & (r \ge R). \end{cases}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{2}{3}\mu_0\sigma R\omega\hat{\mathbf{z}} = \frac{2}{3}\mu_0\sigma R\boldsymbol{\omega}.$$

Surprisingly, the field inside the spherical shell is uniform.

Example 6.1 Find the magnetic field of a uniformly magnetized sphere of radius R.



Sol : Choosing the z axis along the direction of M,

$$\text{we have} \begin{cases} \mathbf{J}_b' = \nabla\times\mathbf{M} = 0 \\ \mathbf{K}_b' = \mathbf{M}\times\hat{\mathbf{n}}' = M\sin\theta\hat{\boldsymbol{\varphi}} \end{cases}$$

The surface current density is analogous to that of a spinning spherical shell with uniform surface current density.

$$\mathbf{K}_b' = \mathbf{M}\times\hat{\mathbf{n}}' = M\sin\theta\hat{\boldsymbol{\varphi}} \Leftrightarrow \mathbf{K}' = \sigma\mathbf{v}' = \sigma R\omega\sin\theta\hat{\boldsymbol{\varphi}}$$

$$\sigma R\omega \to M$$

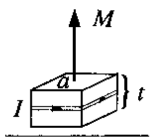
$$\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M} \ \text{(inside)}$$

Can you find a more direct method?

6.2.2 Physical Interpretation of Bound Current

Bound surface current K_b:

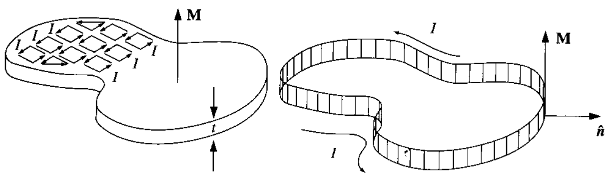
What is the current in terms of M?



In terms of the magnetization M, its dipole moment is $m=Mat=Ia$. So, $M=I/t=K_b$

Consider a thin slab of *uniformly* magnetized material, with the dipoles represented by tiny current loops.

All the “internal” currents cancel. However, at the edge there is no adjacent loop to do the canceling.



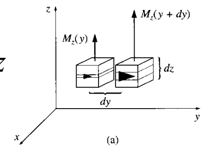
Physical Interpretation of Bound Current

Bound current density J_b:

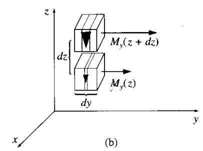
What if the magnetization is not uniform?

The adjacent current loops do not completely cancel out.

Case (a) $I_x = [M_z(y+dy) - M_z(y)]dz = \dfrac{\partial M_z}{\partial y}dydz$



Case (b) $I_x = [M_y(z+dz) - M_y(z)]dy = \dfrac{\partial M_y}{\partial z}dydz$



$$\therefore (J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \Rightarrow \mathbf{J}_b = \nabla\times\mathbf{M}$$

6.3 The Auxiliary Field $\mathbf{H}$
6.3.1 Ampere's Law in Magnetized Materials

What is the difference between bound current and free current?

$$\mathbf{J}=\mathbf{J}_b+\mathbf{J}_f$$

Ampere's law can be written:

$$\frac{1}{\mu_0}(\nabla\times\mathbf{B})=\mathbf{J}=\mathbf{J}_f+\mathbf{J}_b=\mathbf{J}_f+\nabla\times\mathbf{M}$$

$$\Rightarrow\ \nabla\times\underbrace{\left(\frac{1}{\mu_0}\mathbf{B}-\mathbf{M}\right)}_{\mathbf{H}}=\mathbf{J}_f$$

In terms of $\mathbf{H}$, then the Ampere's law reads

$$\nabla\times\mathbf{H}=\mathbf{J}_f\quad\text{(differential form)}$$

$$\oint\mathbf{H}\cdot d\mathbf{l}=\mathbf{I}_f\quad\text{(integral form)}$$

The Role of $\mathbf{H}$ in Magnetostatics

$\mathbf{H}$ plays a role in magnetostatics analogous to $\mathbf{D}$ in the electrostotics.

$\mathbf{D}$ allows us to write Gauss's law in terms of free change alone.

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P},\qquad\nabla\cdot\mathbf{D}=\rho_f$$

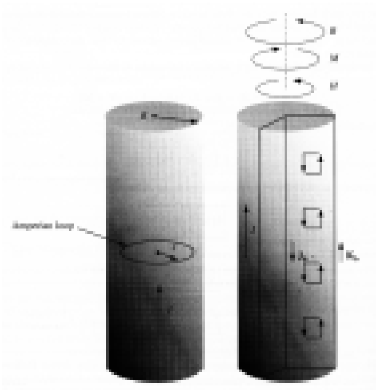
$\mathbf{H}$ permits us to express Ampere's law in terms of free current alone.

$$\mathbf{H}=\frac{1}{\mu_0}\mathbf{B}-\mathbf{M},\qquad\nabla\times\mathbf{H}=\mathbf{J}_f$$

What we can control directly.

Why can't we turn the bound currents on or off independently?

Example 6.2 A long copper rod of radius R carries a uniformly distributed (free) current I. Find $\mathbf{H}$ inside and outside the rod.



How to choose a suitable Amperian loop? Symmetry.

Sol:

Use the Ampere's law in the integral form and properly choose a suitable Amperian loop.

$$s\le R:\quad H(2\pi s)=I_{f_{enc}}=I\frac{\pi s^2}{\pi R^2}$$

$$\text{so}\ \ \mathbf{H}=\frac{sI}{2\pi R^2}\hat{\boldsymbol{\varphi}}$$

$$s>R:\quad H(2\pi s)=I,\quad\text{so}\ \ \mathbf{H}=\frac{I}{2\pi s}\hat{\boldsymbol{\varphi}}$$

How to determine the magnetic field $\mathbf{B}$?

H and B, D and E

$$\nabla \times \mathbf{H} = \mathbf{J}_f$$
$$\nabla \cdot \mathbf{D} = \rho_f$$

Which equation is more useful?

We can easily control the free current **I**, but not the free charge. So **H** can be determined accordingly.

On the other hand, the potential difference **V** can be read from the voltmeter, which can be used to determine **E**.

The name of **H**: Some author call **H**, not **B**, the "magnetic field", but it is not a good choice. Let's just call it "**H**".

6.3.2 A Deceptive Parallel

In free space $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$
$\nabla \cdot \mathbf{B} = 0$

In matter $\nabla \times \mathbf{H} = \mathbf{J}_f$

$$\nabla \cdot \mathbf{H} = \nabla \cdot \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = -\nabla \cdot \mathbf{M} \neq 0$$

At what condition the divergence of **H** is equal to zero?

$\mathbf{M} // \mathbf{B}$ i.e. $\mathbf{M} // \mathbf{B} // \mathbf{H}$ for uniform material only.

6.3.3 Boundary Conditions

The magnetostatic boundary conditions can be rewritten in terms of **H** and the free surface current $\mathbf{K}_f$.

$$\nabla \times \mathbf{H} = \mathbf{J}_f \quad \Rightarrow \quad \mathbf{H}^{//}_{above} - \mathbf{H}^{//}_{below} = \mathbf{K}_f \times \hat{\mathbf{n}}$$
$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \quad \Rightarrow \quad H^{\perp}_{above} - H^{\perp}_{below} = -(M^{\perp}_{above} - M^{\perp}_{below})$$

The corresponding boundary condition in terms of **B** and total surface current **K**.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \Rightarrow \quad \mathbf{B}^{//}_{above} - \mathbf{B}^{//}_{below} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$$
$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad B^{\perp}_{above} - B^{\perp}_{below} = 0$$

How to express the boundary conditions at metal or dielectric interface?

Homework #11

Problems: 6.4, 6.10, 6.13, 6.15

6.4 Linear and Nonlinear Media
6.4.1 Magnetic susceptibility and Permeability

The magnetization of paramagnetic and diamagnetic materials is sustained by the field, i.e. when **B** is removed, **M** disappears.

$$\mathbf{M} = \chi_m \mathbf{H},$$

where the proportionality constant χ_m is called the magnetic susceptibility.

Why not use $\mathbf{M} = \frac{\chi_m}{\mu_0}\mathbf{B}$? Because $\mathbf{M} = \chi_m \mathbf{H} \propto I_f$

Materials that obey $\mathbf{M} = \chi_m \mathbf{H}$ are called linear media.

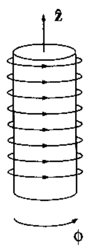
$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H},$$

where $\mu = \mu_0(1 + \chi_m)$ is call the **permeability** of the material.

Material Susceptibility

Material	Susceptibility	Material	Susceptibility
Diamagnetic:		*Paramagnetic:*	
Bismuth	-1.6×10^{-4}	Oxygen	1.9×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminum	2.1×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	7.8×10^{-5}
Water	-9.0×10^{-6}	Platinum	2.8×10^{-4}
Carbon Dioxide	-1.2×10^{-8}	Liquid Oxygen ($-200°$ C)	3.9×10^{-3}
Hydrogen	-2.2×10^{-9}	Gadolinium	4.8×10^{-1}

Table 6.1 Magnetic Susceptibilities (unless otherwise specified, values are for 1 atm, 20° C). *Source: Handbook of Chemistry and Physics,* 67th ed. (Boca Raton: CRC Press, Inc., 1986).



Example 6.3 An infinite solenoid (n turns per unit length, current I) is filled with linear material of susceptibility χ_m: Find the magnetic field inside the solenoid.

Sol: The problem exhibits solenoidal symmetry. Thus, we can employ the Ampere's law.

$$\oint \mathbf{H} \cdot d\mathbf{l} = \mathbf{I}_f \quad \text{(integral form)}$$

$$H\ell = n\ell I \quad \therefore \quad \mathbf{H} = nI\hat{\mathbf{z}}$$

$$\mathbf{B} = \mu_0(1 + \chi_m)nI\hat{\mathbf{z}}$$

The enhancement of the magnetic field strength depends on the susceptibility of the material.

Is there a material that the field is significantly enhanced?

Divergence of the Magnetization

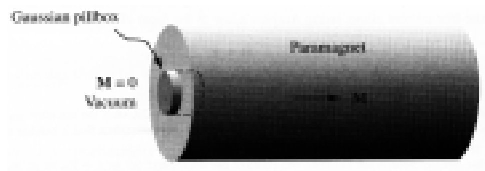
Does the linear media avoid the defect that the divergence of M is zero? **No!**

Even though **M**, **H**, and **B** are parallel, the divergence of **M** is not zero at the boundary. Consider the following example.

$$\oint \mathbf{M} \cdot d\mathbf{a} \neq 0$$

Gaussian pillbox

$$\Rightarrow \quad \nabla \cdot \mathbf{M} \neq 0$$



and $\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times \chi_m \mathbf{H} = \chi_m \mathbf{J}_f$

6.4.2 Ferromagnetism

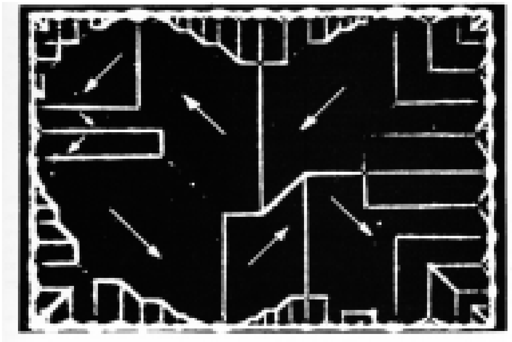
Ferromagnets---which are not linear---require no external fields to sustain the magnetization unlike paramagnets and diamagnets.

In a ferromagnet, each dipole "like" to point in the same direction as its neighbors. All the spins point the same way.

Why isn't every wrench and nail a powerful magnet?

Domains.

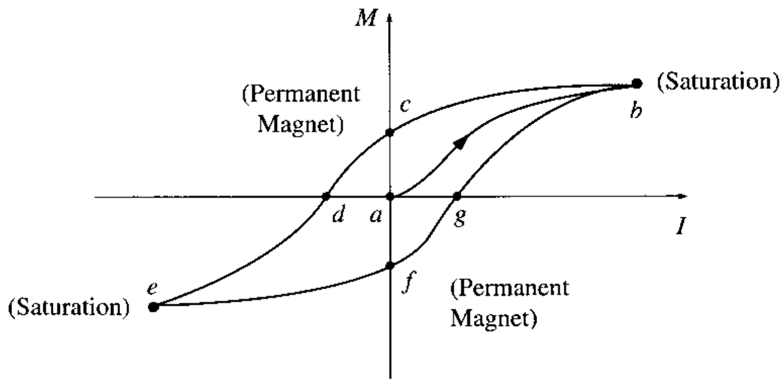
Ferromagnetic Domains



Domain boundaries: Domains parallel to the field grow, and the others shrink.

If the field is strong enough, one domain takes over entirely, and the iron is said to be "**saturated**".

Hysteresis Loop



Hysteresis: The path we have traced out.

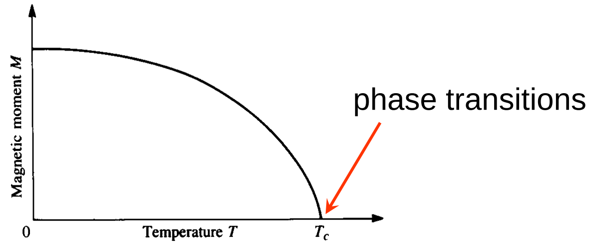
In the experiment, we adjust the current I, i.e. control $\mathbf{H}$.

In practice M is huge compared to H.

Curie Temperature and Phase Transitions

Temperature effect: The dipoles within a given domain line up parallel to one another. However, the random thermal motions complete with this ordering.

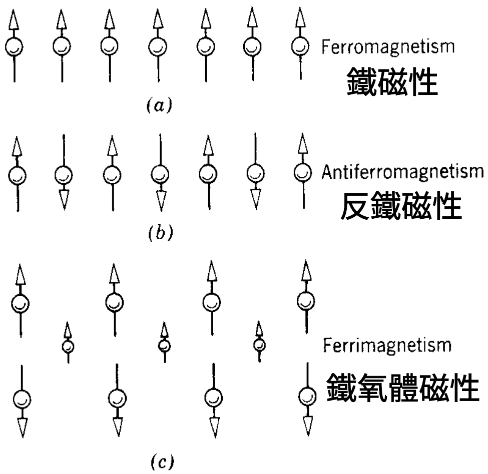
Curie temperature: As the temperature increases, the alignment is gradually destroyed. At certain temperature the iron completely turns into paramagnet. This temperature is called the curie temperature.



Homework #12

Problems: 6.17, 6.21, 6.23, 6.26

Supplementary Material (optional):
Introduction to ferrite materials



The ferrites are crystals having small electric conductivity compared to ferromagnetic materials,

Thus they are useful in high-frequency situations because of the absence of significant eddy current losses.

Properties of ferrite materials (I)

Nonreciprocal electrical property: the transmission coefficient through the device is not the same for different direction of propagation.

Unequal propagation constant: The left and right circularly polarized waves have different propagation constant along the direction of external magnetic field B_0.

Anisotropic magnetic properties: The permeability of the ferrite is not a single scalar quantity, but instead is a tensor, which can be represented as a matrix.

Properties of ferrite materials (II)

Ferrites are **ceramiclike materials** with *specific resistivities* that may be as much as 10^{14} greater than that of metals and with *dielectric constants* around 10 to 15 or greater.

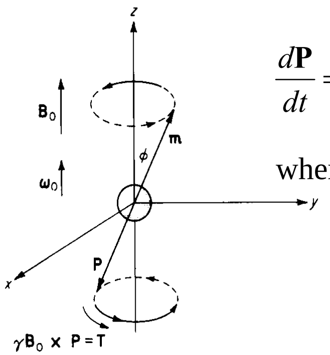
Ferrites are made by sintering **a mixture of metal oxides** and have the general chemical composition $MO \cdot Fe_2O_3$, where M is a divalent metal such as Mn, Mg, Fe, Zn, Ni, Cd, etc.

Relative permeabilities of **several thousand** are common. The magnetic properties of ferrites arise mainly from the **magnetic dipole moment** associated with the **electron spin**.

Classical picture of the magnetization process

--- By treating the spinning electron as a gyroscopic top.

If an electron is located in a uniform static magnetic field $\mathbf{B}_0$, a torque is given by



$$\mathbf{T} = \mathbf{m} \times \mathbf{B_0} = -\frac{m}{p}\mathbf{p} \times \mathbf{B_0} = \frac{e}{m_o}\mathbf{B}_0 \times \mathbf{P}$$

$$\frac{d\mathbf{P}}{dt} = \mathbf{T} = \frac{e}{m_o}\mathbf{B}_0 \times \mathbf{P} = \omega_0 \times \mathbf{P}$$

where $\omega_0 = \frac{eB_0}{m_o}$ is called the Larmor fequency;

$P = \frac{\hbar}{2}$ is angular momentum; and

$m = \frac{e\hbar}{2m_o}$ is magnetic dipole moment.

Quantum mechanics' viewpoint $s_z = \pm 1/2$

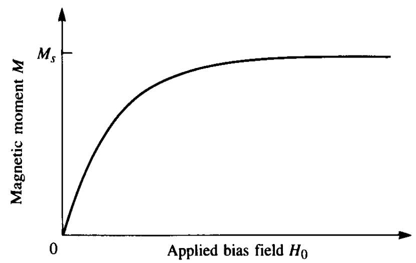
In the absence of any damping forces, the actual precession angle will be determined by the initial position of the magnetic dipole, and the dipole will precess about $\mathbf{B}_0$ at this angle indefinitely (free precession).

In reality, however, the existence of damping forces will cause the magnetic dipole *to spiral in* from its initial angle until **m** is aligned with $\mathbf{B}_0$.

This explains why s_z equals $\pm 1/2$ in the Quantum Mechanics.

But where does the damping force come from?

Saturation magnetization



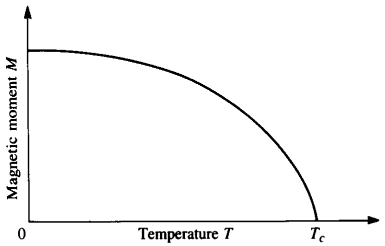
As the strength of the bias field H_0 is increased, more magnetic dipole moments will align with H_0 until all are aligned, and ***M*** reaches an upper limit.

The material is then said to be magnetically saturated, and M_s is denoted as the saturation magnetization. M_s typically ranges from $4\pi M_s$=300 to 5000 Gauss.

Below saturation, ferrite materials can be very lossy at microwave frequencies, and rf interaction is reduced.

The ferrites are usually operated in the saturated state.

Curie temperature



The saturation magnetization of a material is a strong function of temperature, decreasing as temperature increases.

This effect can be understood by noting that the vibrational energy of an atom increases with temperature, making it more difficult to align all the magnetic dipoles.

At a high enough temperature a zero net magnetization results. This temperature is called the Curie temperature, T_c.

Properties of some ferrite materials

Material	Trans-Tech Number	$4\pi Ms$ G	ΔH Oe	ϵ_r	$\tan\delta$	T_c °C	$4\pi Mr$ G
Magnesium ferrite	TT1-105	1750	225	12.2	0.00025	225	1220
Magnesium ferrite	TT1-390	2150	540	12.7	0.00025	320	1288
Magnesium ferrite	TT1-3000	3000	190	12.9	0.0005	240	2000
Nickel ferrite	TT2-101	3000	350	12.8	0.0025	585	1853
Nickel ferrite	TT2-113	500	150	9.0	0.0008	120	140
Nickel ferrite	TT2-125	2100	460	12.6	0.001	560	1426
Lithium ferrite	TT73-1700	1700	<400	16.1	0.0025	460	1139
Lithium ferrite	TT73-2200	2200	<450	15.8	0.0025	520	1474
Yttrium garnet	G-113	1780	45	15.0	0.0002	280	1277
Aluminum garnet	G-610	680	40	14.5	0.0002	185	515

Why use $4\pi M_s$? $\mathbf{B} = 4\pi\mathbf{M} + \mathbf{H} = \mu\mathbf{H}$ (Gaussian unit)

The unit of **B** is gauss; the unit of **H** is Oersted. They have same dimension.

What does ΔH and M_r mean?

Ferrite linewidth and remanent magnetization

Anisotropic magnetic properties (I)

If $\bar{H}$ is the applied ac field, the total magnetic field is $\bar{H}_t = H_0\hat{z} + \bar{H}$,

where $|\bar{H}| << H_0$. The field produced a total magnetization is the ferrite given by $\bar{M}_t = M_s\hat{z} + \bar{M}$.

M_s is the dc saturation magnetization and $\bar{M}$ is the additional ac magnetization (in the xy plane) caused by applied field.

The component equations of motion:

$$\frac{dM_x}{dt} = -\mu_0\gamma M_y(H_0 + H_z) + \mu_0\gamma(M_s + M_z)H_y$$

$$\frac{dM_y}{dt} = -\mu_0\gamma M_x(H_0 + H_z) - \mu_0\gamma(M_s + M_z)H_x$$

$$\frac{dM_z}{dt} = -\mu_0\gamma M_x H_y + \mu_0\gamma M_y H_x$$

Anisotropic magnetic properties (II)

Omitting higher order terms, the equations can be reduced to

$$\left.\begin{aligned}\frac{dM_x}{dt} &= -\omega_0 M_y + \omega_m H_y,\\ \frac{dM_y}{dt} &= -(\omega_0 M_x + \omega_m H_x),\end{aligned}\right\} \Rightarrow \begin{aligned}\frac{d^2M_x}{dt^2} + \omega_0^2 M_x &= \omega_m\frac{dH_y}{dt} + \omega_0\omega_m H_x,\\ \frac{d^2M_y}{dt^2} + \omega_0^2 M_y &= -\omega_m\frac{dH_x}{dt} + \omega_0\omega_m H_y.\end{aligned}$$

$$\frac{dM_z}{dt} = 0, \text{ where } \omega_0 = \mu_0\gamma H_0 \text{ and } \omega_m = \mu_0\gamma M_s$$

If $\bar{M}$ and $\bar{H} \propto e^{j\omega t}$, the above equtions can be reduced to the phasor equations:

$$\begin{aligned}(\omega_0^2 - \omega^2)M_x &= \omega_0\omega_m H_x + j\omega\omega_m H_y,\\ (\omega_0^2 - \omega^2)M_y &= -j\omega\omega_m H_x + \omega_0\omega_m H_y.\end{aligned} \Rightarrow \bar{M} = [\chi]\bar{H} = \begin{bmatrix}\chi_{xx} & \chi_{xy} & 0\\ \chi_{yx} & \chi_{yy} & 0\\ 0 & 0 & 0\end{bmatrix}\bar{H},$$

where $\chi_{xx} = \chi_{yy} = \dfrac{\omega_0\omega_m}{\omega_0^2 - \omega^2}$ and $\chi_{xy} = -\chi_{yx} = \dfrac{j\omega\omega_m}{\omega_0^2 - \omega^2}$

Anisotropic magnetic properties (III)

To relate *B* and *H*, we have

$$\bar{B} = \mu_0(\bar{M} + \bar{H}) = [\mu]\bar{H} \Rightarrow [\mu] = \mu_0([U] + [\chi]) = \begin{bmatrix} \mu & j\kappa & 0 \\ -j\kappa & \mu & 0 \\ 0 & 0 & \mu_0 \end{bmatrix}$$

$$\begin{cases} \mu = \mu_0(1+\chi_{xx}) = \mu_0(1+\chi_{yy}) = \mu_0(1+\dfrac{\omega_0\omega_m}{\omega_0^2 - \omega^2}) \\ \kappa = -j\mu_0\chi_{xy}) = j\mu_0\chi_{yx} = \mu_0\dfrac{\omega\omega_m}{\omega_0^2 - \omega^2}) \end{cases}$$

A material having a permeability tensor of this form is called gyrotropic.

How to apply this concept to a circularly polarized wave?

Forced precession of spinning electron (I)

If a small ac magnetic field is superimposed on the static field $\mathbf{H}_0$, the magnetic dipole moment will undergo a forced precession.

Of particular interest is the case where the ac magnetic field is circularly polarized in the plane perpendicular to $\mathbf{H}_0$.

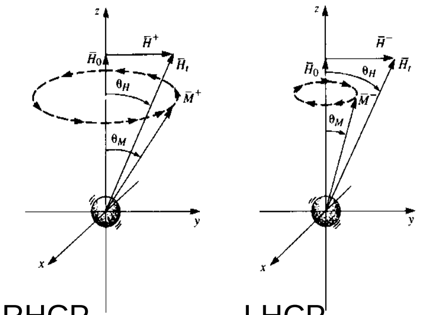
A right-hand circularly polarized wave can be expressed in phasor form as

$$\bar{H}^+ = H^+(\hat{x} - j\hat{y})$$

and in time-domain form as

$$\bar{\mathcal{H}}^+ = \mathrm{Re}\{\bar{H}^+ e^{j\omega t}\} = H^+(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$$

Forced precession of spinning electron (II)



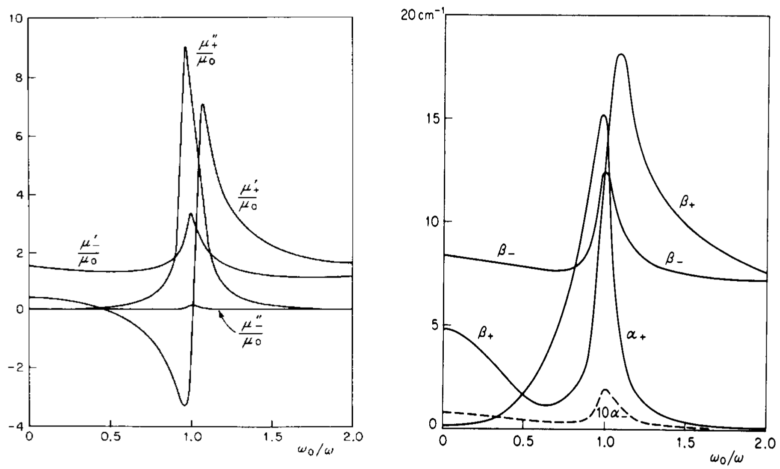
propagation constant

$$\beta_\pm = \omega\sqrt{\varepsilon\mu^\pm}$$

$$\left.\begin{aligned} M_x^+ &= \frac{\omega_m}{\omega_0 - \omega}H^+, \\ M_y^+ &= \frac{-j\omega_m}{\omega_0 - \omega}H^+. \end{aligned}\right\} \Rightarrow \bar{M}^+ = \frac{\omega_m}{\omega_0 - \omega}\bar{H}^+ \quad \Rightarrow \mu^+ = \mu_0(1 + \frac{\omega_m}{\omega_0 - \omega}) \text{ RHCP}$$

$$\mu^- = \mu_0(1 + \frac{\omega_m}{\omega_0 + \omega}) \text{ LHCP}$$

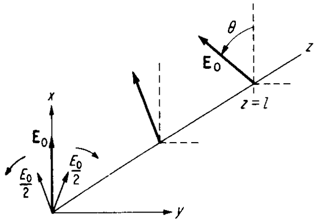
Real and imaginary permeability & propagation and attenuation constant.



Faraday rotation --- a nonreciprocal effect

Consider linearly polarized electric field at z=0, represented as the sum of a RHCP and a LHCP wave:

$$\bar{E}\Big|_{(z=0)} = \hat{x}E_0 = \frac{E_0}{2}(\hat{x} - j\hat{y}) + \frac{E_0}{2}(\hat{x} + j\hat{y})$$



These two polarized waves propagate with different propagation constants.

$$\bar{E}\Big|_{(z=\ell)} = \frac{E_0}{2}(\hat{x} - j\hat{y})e^{-j\beta_+\ell} + \frac{E_0}{2}(\hat{x} + j\hat{y})e^{-j\beta_-\ell}$$

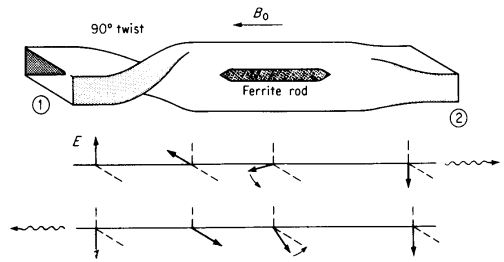
$$= E_0\left[\hat{x}\cos\left(\frac{\beta_+ - \beta_-}{2}\right)\ell - \hat{y}\sin\left(\frac{\beta_+ - \beta_-}{2}\right)\ell\right]e^{-j(\beta_+ + \beta_-)\ell/2}$$

$$\theta = \tan^{-1}\frac{E_y}{E_x} = -\left(\frac{\beta_+ - \beta_-}{2}\right)\ell.$$

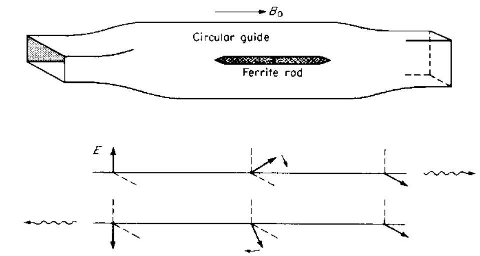
This effect is called *Faraday rotation*.

Microwave gyrator

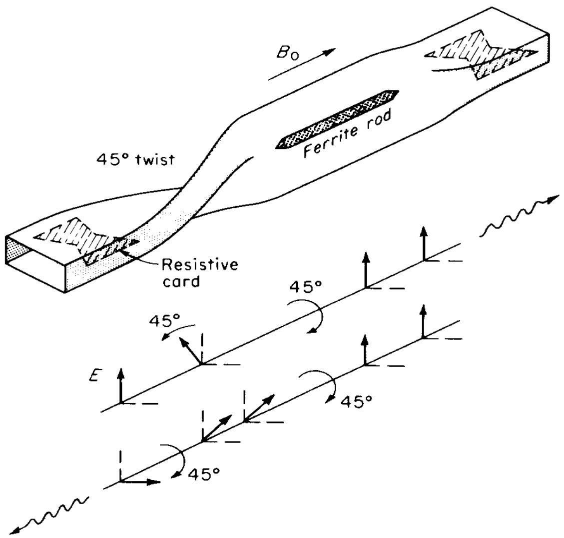
Gyrator with a twist section.



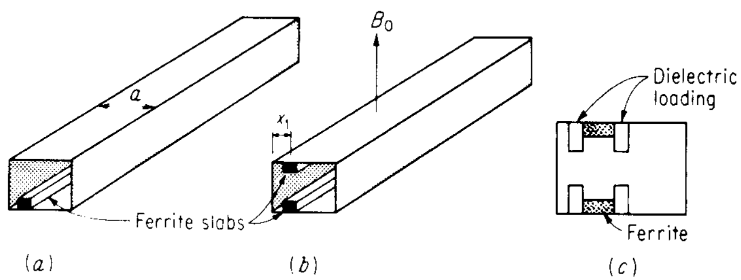
Gyrator without a twist section.



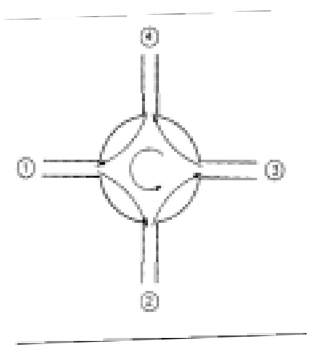
Faraday-rotation isolator

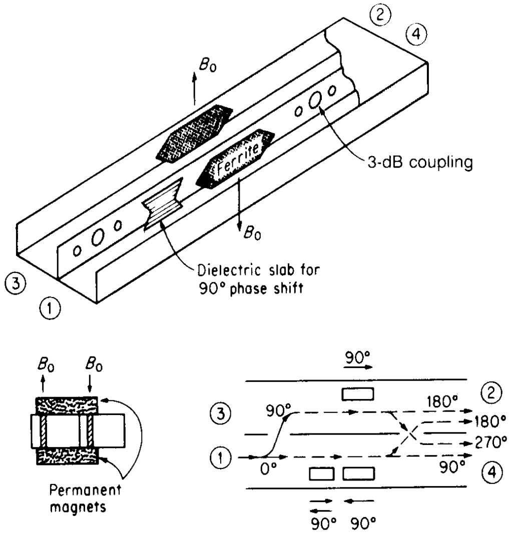


Resonance isolator

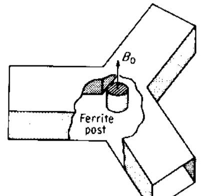


Four-port circulator

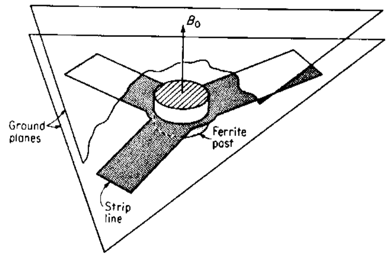




Three-port circulator



(a)



(b)