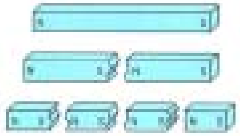
Chapter 5 Magnetostatics

5.1 The Lorentz Force Law 5.1.1 Magnetic Fields

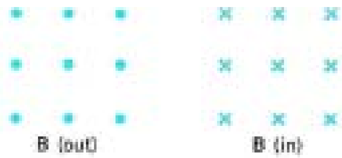
By analogy with electrostatics, why don't we study magnetostatics first? *Due to lack of **magnetic monopole**.*



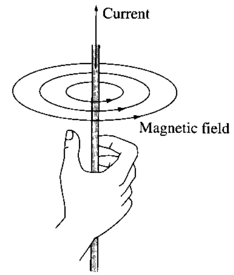
If one try to isolate the poles by cutting the magnet, a curious thing happens: One obtains two magnets. No matter how thinly the magnet is sliced, each fragment always have two poles. Even down to the atomic level, no one has found an isolated magnetic pole, called a monopole. Thus magnetic field lines form closed loops.

The Magnetic Field

Outside a magnetic the lines emerge from the north pole and enter the south pole; *within* the magnet they are directed from the south pole to the north pole. The ***dots*** represents the tip of an arrow coming toward you. The ***cross*** represents the tail of an arrow moving away.

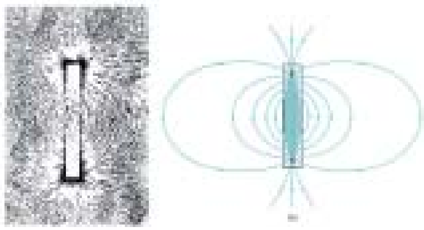


How a current-carry wire produces a magnetic field?



The Magnetic Field of a Bar Magnet

When iron filings are sprinkled around a bar magnet, they form a characteristic pattern that shows how the influence of the magnet spreads to the surrounding space.



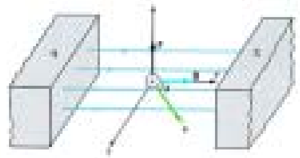
The **magnetic field**, $\mathbf{B}$, at a point along the tangent to a field line. The *direction* of $\mathbf{B}$ is that of the force on the north pole of a bar magnet, or the *direction* in which a compass needle points. The *strength* of the field is proportional to the number of lines passing through a unit area normal to the field (*flux density*).

Definition of the Magnetic Field

When defining of the electric field, the electric field strength can be derived from the following relation: $\mathbf{E}=\mathbf{F}/q$. Since an isolated pole is not available, *the definition of the magnetic field is not as simple*.

Instead, we examine how an electric charge is affected by a magnetic field.

$$\mathbf{F} = qvB\sin\theta = q\mathbf{v}\times\mathbf{B}$$



Since $\mathbf{F}$ is always perpendicular to $\mathbf{v}$, *a magnetic force does no work on a particle and cannot be used to change its kinetic energy.*

The SI unit of magnetic field is the Tesla (T). 1 T=10^4 G

The Lorentz Force Law

When a particle is subject to both electric and magnetic fields in the same region, what is the total force on it?

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

This is called the **Lorentz force law**. This axiom is found in experiments.

The magnetic force do no work.

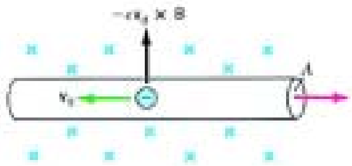
$$dW_{mag} = \mathbf{F}_{mag} \cdot d\mathbf{l} = q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}dt = 0$$

Really? But, how do you explain a magnetic crane lifts a container?

Force on a Current-Carrying Conductor

When a current flows in a magnetic field, the electrons as a whole acquire a slow drift speed, $\mathbf{v}_d$, and experience a magnetic force, which is then transmitted to the wire.

$$\begin{aligned} F &= qvB\sin\theta = (nA\ell e)v_d B \\ &= (nAev_d)\ell B \\ &= I\ell B \end{aligned}$$



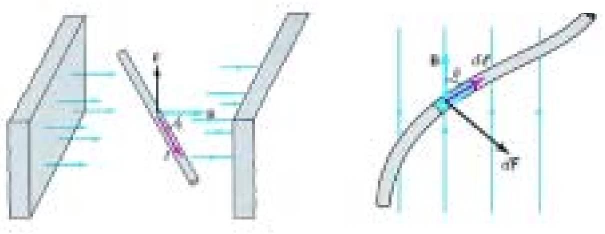
$$\mathbf{F} = I\boldsymbol{\ell} \times \mathbf{B}$$
$$F = I\ell B\sin\theta$$

n: is the number of the conductor per unit volume.
ℓ: is defined to be in the direction in which the current is flowing.

Force on a Current-Carrying Conductor

The force on an infinitesimal current element is

$$d\mathbf{F} = Id\boldsymbol{\ell} \times \mathbf{B}$$



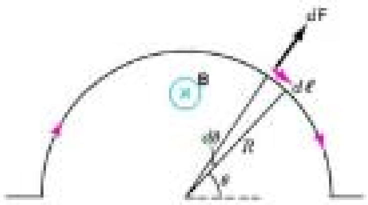
The force on a wire is the vector sum (integral) of the forces on all current elements.

Example:
The Magnetic Force on a Semicircular Loop

A wire is bent into a semicircular loop of radius R. It carries a current I, and its plane is perpendicular to a uniform magnetic field $\mathbf{B}$, as shown below. Find the force on the loop.

Solution:

$$d\mathbf{F} = Id\boldsymbol{\ell} \times \mathbf{B}$$
$$dF_y = IRB\sin\theta\ d\theta$$
$$\begin{aligned} F_y &= IRB\int_0^{\pi}\sin\theta\ d\theta \\ &= 2IRB = I(2R)B \end{aligned}$$



The x-components of the forces on such elements will cancel in pairs.

The net force on any close current-carrying loop is zero.

The Motion of Charged Particles in Magnetic Fields

How does a charged particle move with an initial velocity **v** perpendicular to a uniform magnetic field **B**?

Since **v** and **B** are perpendicular, the particle experiences a force $F=qvB$ of constant magnitude directed perpendicular. Under the action of such a force, the particle will move in a circular path at constant speed. From Newton's second law, $F=ma$, we have

$$qvB = \frac{mv^2}{r} \quad \Rightarrow \quad r = \frac{mv}{qB}$$

The radius of the orbit is directly proportional to the linear momentum of the particle and inversely proportional to the magnetic field strength.

Cyclotron Motion

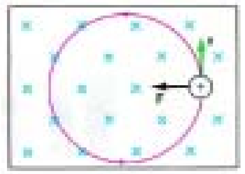
What are the frequency and the period? Are they independent of the speed of the particle? *Yes*.

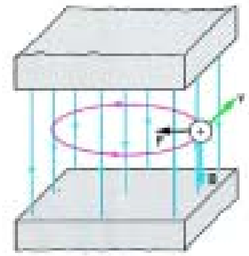
The period of the orbit is

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{qB} = \left(\frac{m}{q}\right)\frac{2\pi}{B}$$

$$f_c = \frac{1}{T} = \frac{qB}{2\pi m} = \left(\frac{q}{m}\right)\frac{B}{2\pi}$$

$$\boxed{f_c/B = 2.8\ \text{MHz/Gauss}}$$





The frequency is called the cyclotron frequency.

All particles with the same charge-to-mass ratio, q/m, have the same period and cyclotron frequency.

Example: Cyclotron

A cyclotron is used to accelerate protons from rest. It has a radius of 60 cm and a magnetic field of 0.8 T. The potential difference across the dees is 75 kV. Find: (a) the frequency of the alternating potential difference; (b) the maximum kinetic energy; (c) the number of revolutions made by the protons.

Solution:

(a) $f_c = \frac{qB}{2\pi m} = 12\ \text{MHz}$

(b) $K_{\max} = \frac{(qr_{\max}B)^2}{2m} = 1.76\times10^{-12}\ \text{J} = 11\ \text{MeV}$

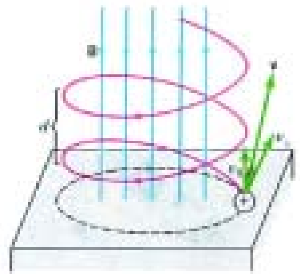
(c) $\Delta K = 2qV = 150\ \text{keV}$

$K_{\max}/\Delta K = 11000/150 = 73.5\ \text{revs.}$

How to determine the maximum kinetic energy?

Helical Motion

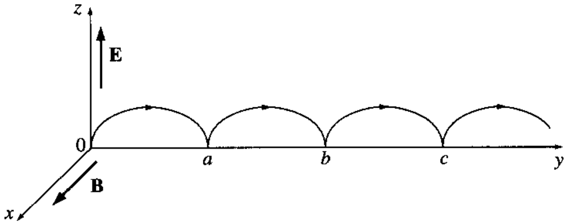
What happens if the charged particle's velocity has not only a perpendicular component $v_\perp$ but also a parallel component $v_{//}$? *Helical Motion*.



The perpendicular component $v_\perp$ gives rise to a force $qv_\perp B$ that produces circular motion, but the parallel component $v_{//}$ is not affected. The result is the superposition of a uniform circular motion normal to the lines and a constant motion along the lines.

Example: Cycloid Motion

Suppose, for instance, that **B** points in the x-direction, and **E** in the z-direction. A particle initially at rest is released from the origin; what path will it follow?

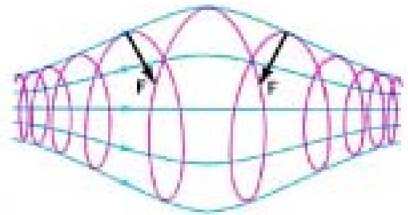


Solution:

1. Write down the equation of motion.
2. Solve the coupled differential equations.
3. Determine the constants using the initial conditions.

Magnetic Bottle/Mirror

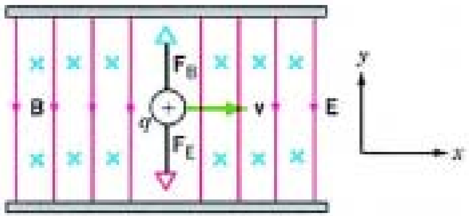
What happens if the magnetic field is not uniform? Energy transfer between the perpendicular and parallel components.



In a nonuniform field, the particle experiences a force that points toward the region of week field. As a result, the component of the velocity along the **B** lines is not constant.

If the particle is moving toward the region of stronger field, as some point it may be stopped and made to reverse the direction of its travel.

Velocity Selector

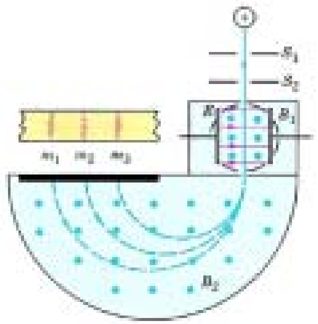


$$\begin{pmatrix} \mathbf{E} = -E\mathbf{j} \\ \mathbf{B} = -B\mathbf{k} \end{pmatrix} \Rightarrow \mathbf{v} = \frac{E}{B}\mathbf{i} = v\mathbf{i}$$

Only those particles with speed $v=E/B$ pass through the crossed fields undeflected. This provides a convenient way of either measuring or selecting the velocities of charged particles.

Mass Spectrometer

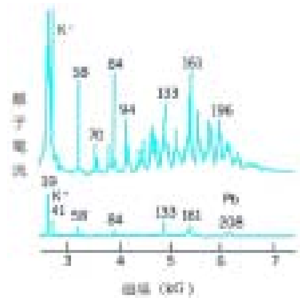
A mass spectrometer is a device that separates charged particles, usually ions, according to their charge-to-mass ratios.



$$v = \frac{E}{B_1}$$

$$r = \frac{mv}{qB_2} = \frac{mE}{qB_1B_2}$$

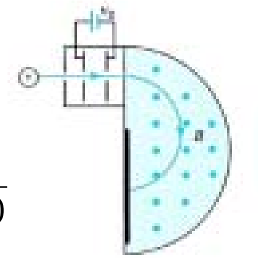
$$\frac{m}{q} = \frac{B_1B_2}{E}r$$



Example: Mass Spectrometer

In a mass spectrometer shown below, two isotopes of an elements with mass $m1$ and $m2$ are accelerated from rest by a potential difference V. They then enter a uniform $\mathbf{B}$ normal to the magnetic field lines. What is the ratio of the radii of their paths?

Solution:



$$v = \sqrt{\frac{2qV}{m}}$$

$$r = \frac{mv}{qB} = \sqrt{\frac{2mV}{qB^2}} \quad \text{then } r_1 / r_2 = \sqrt{(m_1 / m_2)}$$

Note1: How particle is accelerated by a potential difference?

Current and Surface Current

The **current** in a wire is the charge per unit time passing a given point.

Current is measured in coulombs-per-second, or **amperes** (A).

1 A=1 C/s

The **surface current density, $\mathbf{K}$,** is defined as follows: consider a "ribbon" of infinitesimal width $d\ell_\perp$, running parallel to the flow. Then,

$$\mathbf{K} = \frac{d\mathbf{I}}{d\ell_\perp}$$

In words, $\mathbf{K}$ is the current per unit width-perpendicular-to-flow.

$$\mathbf{K} = \frac{d\mathbf{I}}{d\ell_\perp} \approx \frac{d\left(\frac{d(\sigma \ell_\perp \ell_{//})}{dt}\right)}{d\ell_\perp} = \sigma \frac{d\ell_{//}}{dt} = \sigma \mathbf{v}$$

Volume Current Density

The **volume current density, $\mathbf{J}$,** is defined as follows: consider a "tube" of infinitesimal cross section $da_\perp$, running parallel to the flow. Then,

$$\mathbf{J} = \frac{d\mathbf{I}}{da_\perp}$$

In words, $\mathbf{J}$ is the current per unit area-perpendicular-to-flow.

$$\mathbf{J} = \frac{d\mathbf{I}}{da_\perp} \approx \frac{d\left(\frac{d(\rho a_\perp \ell_{//})}{dt}\right)}{da_\perp} = \rho \frac{d\ell_{//}}{dt} = \rho \mathbf{v}$$

Conservation of Charge

The current crossing a surface S can be written as

$$I = \int_S \mathbf{J} \cdot \hat{\mathbf{n}} da$$

In particular, the total charge per unit time leaving a volume V is

$$I = \oint_S \mathbf{J} \cdot \hat{\mathbf{n}} da = \int_V (\nabla \cdot \mathbf{J}) d\tau = -\frac{dQ}{dt}$$

$$\text{where } \frac{dQ}{dt} = \frac{d}{dt} \int_V \rho d\tau.$$

$$\int_V (\nabla \cdot \mathbf{J}) d\tau = -\frac{d}{dt} \int_V \rho d\tau \quad \Rightarrow \boxed{\nabla \cdot \mathbf{J} = -\frac{d\rho}{dt}}$$

continuity equation

5.2 The Biot-Savart Law 5.2.1 Steady Currents

Stationary charges produce electric fields that are constant in time. *Steady currents* produce magnetic fields that are also constant in time.

Stationary charges $\Rightarrow$ constant electric fields; electrostatics. Steady currents $\Rightarrow$ constant magnetic fields; magnetostatics.

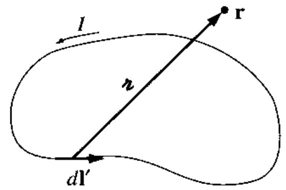
Steady current means that a continuous flow that goes on forever without change and without charge piling up anywhere. They represent *suitable approximations* as long as the fluctuations are reasonably slow.

$$\nabla \cdot \mathbf{J} = 0$$

5.2.2 The Magnetic Field of a Steady Current

The **Biot-Sarvart law**:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

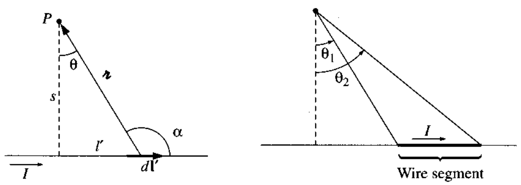


The integration is along the current path, in the direction of the flow.
μ_0: the *permeability of free space*.

Definition of magnetic field $\mathbf{B}$: newtons per ampere-meter or **tesla** (T). 1 T=1 N/(A·m)

The Biot-Sarvart law plays a role analogous to Coulomb's law in electrostatics.

Example 5.5 Find the magnetic field a distance s from a long straight wire carrying a steady current I.



Sol :

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Then, determine the suitable coordinate: cylindrical coordinate (s, ϕ, z).

In the diagram, $(d\mathbf{l}'\times\hat{\mathscr{r}})$ points out of page and have the magnitude

$$dl'\sin\alpha = dl'\cos\theta$$

$$l' = s\tan\theta \Rightarrow dl' = s\sec^2\theta\, d\theta \quad \text{and} \quad \frac{1}{\mathscr{r}} = \frac{\cos\theta}{s}$$

Con't

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2\theta}{s^2}\frac{s}{\cos^2\theta}\cos\theta\, d\theta$$

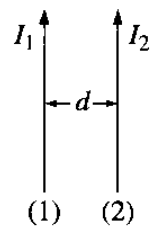
$$= \frac{\mu_0 I}{4\pi s}\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\mu_0 I}{2\pi s}\left(= 2\times 10^{-7}\frac{I}{s}\ \text{Tesla}\right)$$

What is the force between two parallel current-carrying wires?

$$d\mathbf{F} = I d\mathbf{l}\times\mathbf{B}$$

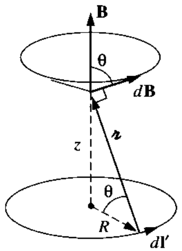
$$dF = I_2\frac{\mu_0 I_1}{2\pi d}dl = \frac{\mu_0 I_1 I_2}{2\pi d}dl$$

$$\frac{dF}{dl} = \frac{\mu_0 I_1 I_2}{2\pi d}$$



(attractive force per unit length, why?)

Example 5.6 Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I.



Sol : $\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}' \times \hat{\mathbf{r}}}{r^2}$

Choose cylindrical coordinate (s, ϕ, z).

In the diagram, $(d\mathbf{l}' \times \hat{\mathbf{r}})$ sweeps around the z axis, thus only the z-component survives.

z-component of $(d\mathbf{l}' \times \hat{\mathbf{r}}) = dl' \cos\theta = R\cos\theta d\phi$

$$\frac{1}{r^2} = \frac{1}{(R^2+z^2)} \quad \text{and} \quad \sin\theta = \frac{R}{(R^2+z^2)^{1/2}}$$

The Bio-Sarvart Law for the Surface and Volume Current

The **Biot-Sarvart law**: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I} \times \hat{\mathbf{r}}}{r^2} dl' = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}' \times \hat{\mathbf{r}}}{r^2}$

For surface current: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K} \times \hat{\mathbf{r}}}{r^2} da'$

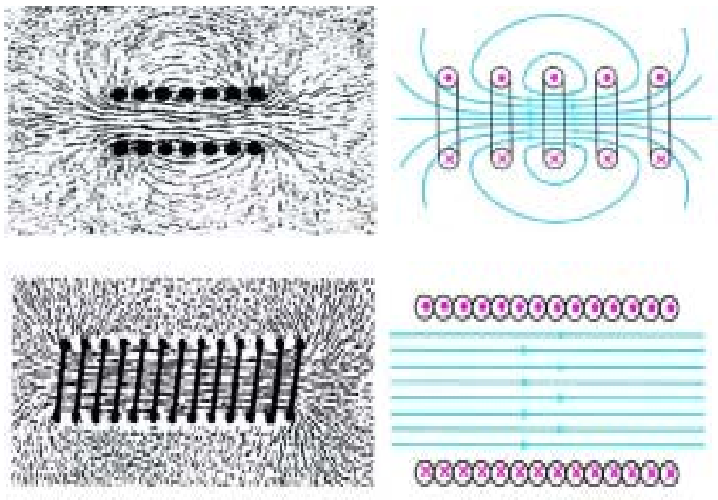
For volume current: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{r^2} d\tau'$

For a moving charge: Wrong, why?

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{r^2} d\tau' = \frac{\mu_0}{4\pi}\int \frac{q\mathbf{v}\delta(\mathbf{r}-\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} d\tau' = \frac{\mu_0}{4\pi}\frac{q\mathbf{v} \times \hat{\mathbf{r}}}{r^2}$$

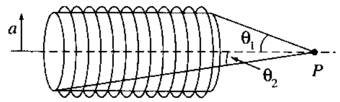
A point charge does not constitute a steady current.

The Magnetic Field of Solenoid



Solenoid

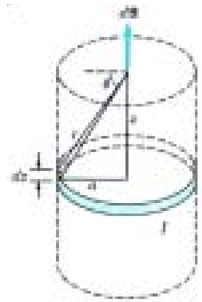
Problem 5.11 A solenoid of length L and radius a has N turns of wire and carries a current I. Find the field strength at a point along the axis.



Solution:

Sine the solenoid is a series of closely packed loops, we may divided into current loops of width dz, each of which contains ndz turns, where $n = N/L$ is the number of turns per unit length.

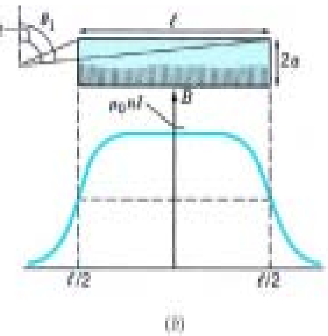
The current within such a loop is $(ndz)I$.



Solenoid (II)

Con't

$$z = a\tan\theta \Rightarrow dz = a\sec^2\theta d\theta$$
$$nIdz = nIa\sec^2\theta d\theta$$



(b)

$$dB_{axis} = \frac{\mu_0 a^2}{2(a^2 + a\tan^2\theta)^{3/2}} nIa\sec^2 d\theta$$
$$= \frac{1}{2}\mu_0 nI\cos\theta d\theta$$
$$B = \int_{\theta_1}^{\theta_2} \frac{1}{2}\mu_0 nI\cos\theta d\theta$$
$$= \frac{1}{2}\mu_0 nI(\sin\theta_2 - \sin\theta_1)$$

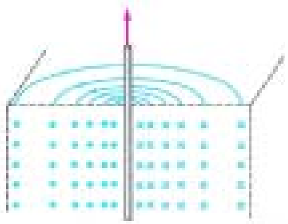
$$B = \mu_0 nI \text{(infinite long solenoid)}$$

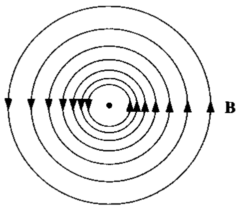
Homework #9

Problems: 9, 10, 11, 39, 49

5.3 The Divergence and Curl of B
5.3.1 Straight-Line Currents

The magnetic field of an infinite straight wire:





$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\varphi}}$$

The integral of **B** around a *circular path* of radius s, centered at the wire, is:

$$\oint \mathbf{B}(\mathbf{r})\cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\varphi}} s\, d\phi = \mu_0 I$$

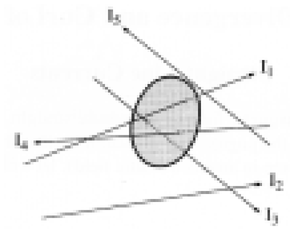
In fact for any loop that encloses the wire would give the same answer. Really?

The Differential Form of B

Suppose we have a bundle of straight wires. Only wires that pass through the loop contribute $\mu_0 I$.
The line integration then be

$$\oint \mathbf{B}(\mathbf{r})\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad I_{\text{enc}} = \int \mathbf{J}\cdot d\mathbf{a}$$

The total current enclosed by the integration loop.

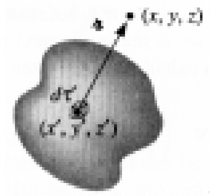


$$\oint \mathbf{B}\cdot d\mathbf{l} = \int(\nabla\times\mathbf{B})\cdot d\mathbf{a} = \int \mu_0 \mathbf{J}\cdot d\mathbf{a}$$
$$\nabla\times\mathbf{B} = \mu_0 \mathbf{J}$$

Does this differential equation apply to any shape of the current loop? Yes, to be prove next.

5.3.2 The Divergence and Curl of **B**

The Biot-Sarvart Law for the general case of a volume current:



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

B is a function of (x, y, z),

J is a function of (x', y', z'),

$$\boldsymbol{\mathscr{r}} = (x-x')\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} + (z-z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

The integration is over the *primed* coordinates.

The divergence and the curl are to be taken with respective to the *unprimed* coordinates.

The Divergence of **B**

The divergence of B:

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = \nabla\cdot\left(\frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'\right) = \frac{\mu_0}{4\pi}\int \nabla\cdot\left(\frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$$\nabla\cdot\left(\frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot(\nabla\times\mathbf{J}) - \mathbf{J}\cdot\left(\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = -\nabla\left(\frac{1}{\mathscr{r}}\right) \quad \text{(Prob. 1.13)}$$

$$\nabla\cdot\left(\frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\underbrace{(\nabla\times\mathbf{J})}_{0} - \mathbf{J}\cdot\underbrace{\left(\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}_{0}$$

$\therefore \underline{\nabla\cdot\mathbf{B} = 0}$ The divergence of a magnetic field is zero.

The Curl of **B**

The curl of B:

$$\nabla\times\mathbf{B} = \frac{\mu_0}{4\pi}\int \nabla\times\left(\frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \underbrace{(\mathbf{B}\cdot\nabla)\mathbf{A}}_{0} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \underbrace{\mathbf{B}(\nabla\cdot\mathbf{A})}_{0}$$

unprimed *primed* *primed +unprimed*

0 to be seen next

$$\nabla\times\left(\frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \mathbf{J}\left(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) - \cancel{(\mathbf{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}$$

$$\nabla\times\left(\frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \mathbf{J}\left(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \mathbf{J}\,4\pi\delta^3(\boldsymbol{\mathscr{r}}) \quad \text{(See 1.5.3)}$$

$$\therefore \nabla\times\mathbf{B} = \frac{\mu_0}{4\pi}4\pi\int \mathbf{J}(\mathbf{r}')\delta^3(\boldsymbol{\mathscr{r}})\,d\tau' = \mu_0\mathbf{J}(\mathbf{r})$$

$\underline{\nabla\times\mathbf{B} = \mu_0\mathbf{J}}$ The curl of B equals μ_0 times **J**.

A Special Technique

$$\int(\mathbf{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' = 0$$

Let's prove that this integration is zero.

special technique $\left\{\begin{array}{l}(\mathbf{J}\cdot\nabla)\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2} = -(\mathbf{J}\cdot\nabla')\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2}, \\ \text{where } \boldsymbol{\mathscr{r}} = (\mathbf{r}-\mathbf{r}')\end{array}\right.$

$$\nabla\cdot(f\mathbf{A}) = \nabla f\cdot\mathbf{A} + f(\nabla\cdot\mathbf{A})$$

Using the above rule, the x component is:

$$(\mathbf{J}\cdot\nabla')\frac{x\text{-}x'}{\mathscr{r}^3} = \nabla'\cdot\left(\frac{x\text{-}x'}{\mathscr{r}^3}\mathbf{J}\right) - \frac{x\text{-}x'}{\mathscr{r}^3}\underbrace{(\nabla'\cdot\mathbf{J})}_{\text{0, for steady current}}$$

$$\int(\mathbf{J}\cdot\nabla)\frac{\hat{\mathscr{r}}_x}{\mathscr{r}^2}\,d\tau' = \int\nabla'\cdot\left(\frac{x-x'}{\mathscr{r}^3}\mathbf{J}\right)d\tau' = \oint_S \underbrace{\left(\frac{x-x'}{\mathscr{r}^3}\mathbf{J}\right)}_{\text{0, since } \mathbf{J}(\mathbf{r}'@\infty)=0}\cdot d\mathbf{a}' = 0$$

What happens if **J**(**r'**)≠0

5.3.3 Applications of Ampere's Law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \text{Ampere's law in differential form}$$

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint_{\text{amperian loop}} \mathbf{B} \times d\mathbf{l} = \int \mu_0 \mathbf{J} \cdot d\mathbf{a} = \mu_0 I_{\text{enc}}$$

$$\oint_{\text{amperian loop}} \mathbf{B} \times d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad \text{Ampere's law in integral form}$$

Just as the Biot-Savart law plays a role in magnetostatics that coulomb's law assumed in electrostatics, so Ampere's play the role of Gauss's.

Electrostatics:	Coulomb	→ Gauss,
Magnetostatics:	Biot-Savart	→ Ampere.

Applications of Ampere's Law

Like Gauss's law, ampere's law is always *true* (for steady currents), but is *not always useful*.

Only when the symmetry of the problem enables you to pull **B** outside the integral can you calculate the magnetic field from the Ampere's law.

These symmetries are:

1. Infinite straight lines
2. Infinite planes (Ex. 5.8)
3. Infinite solenoids (Ex. 5.9)
4. Toroids (Ex. 5.10)

Infinite Straight Wire

Example An infinite straight wire of radius R carries a current I. Find the magnetic field at a distance r from the center of the wire for (a) $r>R$, and (b) $r<R$. Assume that the current is uniformly distributed across the cross section of the wire.

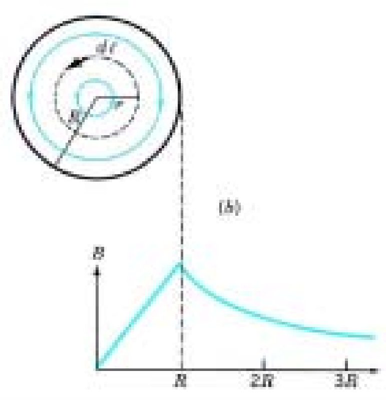
Solution:

(a) $$\oint \mathbf{B} \cdot d\ell = B 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r} \quad (r > R)$$

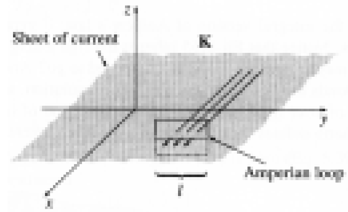
(b) $$\oint \mathbf{B} \cdot d\ell = B 2\pi r = \mu_0 \frac{\pi r^2}{\pi R^2} I$$

$$B = \frac{\mu_0 I}{2\pi R^2} r \quad (r < R)$$



Infinite Planes

Example 5.8 Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K\hat{\mathbf{x}}$, flowing over the xy plane.



Solution:

$$\oint \mathbf{B} \cdot d\ell = B 2l = \mu_0 K l$$

$$B = \frac{\mu_0 K}{2}$$

$$\mathbf{B} = \begin{cases} \mu_0 K / 2\hat{\mathbf{y}} & \text{for } z < 0 \\ -\mu_0 K / 2\hat{\mathbf{y}} & \text{for } z > 0 \end{cases}$$

Solenoid

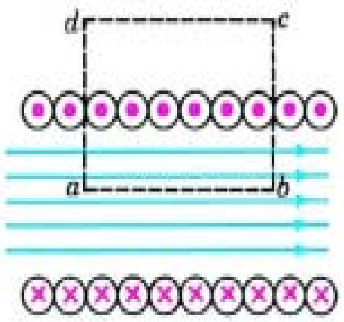
Example 5.9 An ideal infinite solenoid has n turns per unit length and carries a current I. Find its magnetic field inside.

Solution:

$$\oint \mathbf{B}\cdot d\ell$$
$$= \int_a^b B\cdot d\ell + \cancel{\int_b^c B\cdot d\ell} + \cancel{\int_c^d B\cdot d\ell} + \cancel{\int_d^a B\cdot d\ell}$$
$$= \int_a^b B\cdot d\ell$$

$$BL_{ab} = \mu_0 n L_{ab} I$$

$$B = \mu_0 n I$$



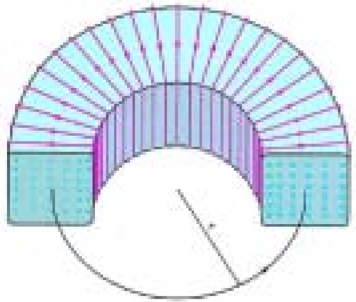
Toroid

Example 5.10 A toroidal coil (shaped like a doughnut) is tightly wound with N turns and carries a current I. We assume that it has a rectangular cross section, as shown below. Find the field strength within the toroid.

Solution:

$$\oint \mathbf{B}\cdot d\ell = B\oint d\ell = \mu_0 NI$$

$$B = \frac{\mu_0 NI}{2\pi r}$$



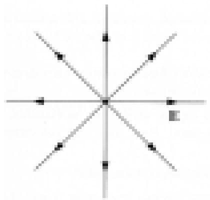
The field is not uniform; it varies as $1/r$. The toroidal fields are used in research on fusion power.

5.3.4 Comparison of Magnetostatics and Electrostatics

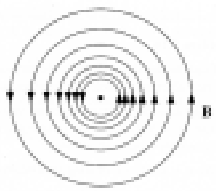
$$\begin{cases} \nabla\cdot\mathbf{E} = \dfrac{\rho}{\varepsilon_0} & \text{Gauss's law} \\ \nabla\times\mathbf{E} = 0 & \text{no name (Faraday's law)} \end{cases}$$

$$\begin{cases} \nabla\cdot\mathbf{B} = 0 & \text{Gauss's law for magnetic field} \\ \nabla\times\mathbf{B} = \mu_0\mathbf{J} & \text{Ampere's law (Ampere-Maxwell law)} \end{cases}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \quad \text{Lorentz's force law}$$



(a) Electrostatic field of a point charge



(b) Magnetostatic field of a long wire

5.4 Magnetic Vector Potential
5.4.1 The Vector Potential

$$\nabla\times\mathbf{E} = 0 \Leftrightarrow E = -\nabla V \quad \text{and} \quad \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\nabla\cdot\mathbf{B} = 0 \Leftrightarrow \mathbf{B} = \nabla\times\mathbf{A} \quad \text{and} \quad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \Rightarrow \nabla(\cancel{\nabla\cdot\mathbf{A}}) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

Is it possible for us to set $\nabla\cdot\mathbf{A} = 0$ equals zero? Yes.

Coulomb gauge

Proof: If $\nabla\cdot\mathbf{A}_0 \neq 0$, let $\mathbf{A} = \mathbf{A}_0 + \nabla\lambda \Rightarrow \mathbf{B} = \nabla\times\mathbf{A}_0 = \nabla\times\mathbf{A}$

If $\nabla\cdot\mathbf{A} = 0$, then $\nabla^2\lambda = -\nabla\cdot\mathbf{A}_0$ $\leftarrow$ similiar to Poisson's equation

$$\begin{cases} \nabla^2 V = -\rho/\varepsilon_0 & V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\rho}{r}\, d\tau' \\ \nabla^2\lambda = -\nabla\cdot\mathbf{A}_0 & \lambda = \dfrac{1}{4\pi}\displaystyle\int \frac{\nabla\cdot\mathbf{A}_0}{r}\, d\tau' \end{cases}$$

It is always possible to make the vector potential divergenceless.

The Vector Potential and Scalar Potential

Using the Coulomb gauge, we obtain: $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathrm{r}}\, d\tau'$$

For line and surface current,

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{\mathrm{r}}\, dl' \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{K(\mathbf{r}')}{\mathrm{r}}\, da'$$

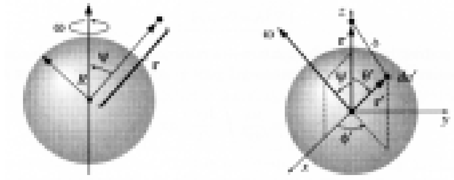
What happens when the curl of **B** vanishes?

Magnetostatic scalar potential.

$$\nabla \times \mathbf{B} = 0 \;\Rightarrow\; \mathbf{B} = -\nabla U$$

$$\Rightarrow \nabla^2 U = 0 \;\text{ (similiar to Laplace's equation)}$$

Example 5.11 A spherical shell, of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω. Find the vector potential it produce at point **r**.



Sol :First, let the observer is in the z axis and ω is tilted at an angle ψ

Vector potential is $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{\mathrm{r}}\, da'$

The surface current density $\mathbf{K}(\mathbf{r}') = \sigma \mathbf{v}'$

$$\mathbf{v}' = \boldsymbol{\omega} \times \mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega[-(\cos\psi\sin\theta'\sin\phi')\,\hat{\mathbf{x}} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\,\hat{\mathbf{y}} + (\sin\psi\sin\theta'\sin\phi')\,\hat{\mathbf{z}}].$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{R\omega(-\cos\psi\sin\theta'\sin\phi')\hat{\mathbf{x}}}{\sqrt{r^2+R^2-2rR\cos\theta'}} R^2\sin\theta' d\theta' d\phi'$$
$$+\frac{\mu_0}{4\pi}\int \frac{R\omega(\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\hat{\mathbf{y}}}{\sqrt{r^2+R^2-2rR\cos\theta'}} R^2\sin\theta' d\theta' d\phi'$$
$$+\frac{\mu_0}{4\pi}\int \frac{R\omega(\sin\psi\sin\theta'\sin\phi')\hat{\mathbf{z}}}{\sqrt{r^2+R^2-2rR\cos\theta'}} R^2\sin\theta' d\theta' d\phi'$$

$$\mathbf{A}(\mathbf{r}) = \frac{-R^3\sigma\omega\sin\psi\mu_0\hat{\mathbf{y}}}{4\pi}\int \frac{\cos\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}}\sin\theta' d\theta' d\phi'$$
$$= \frac{-R^3\sigma\omega\sin\psi\mu_0\hat{\mathbf{y}}}{4\pi}(2\pi)\int_0^{\pi} \frac{-\cos\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}}\, d\cos\theta'$$
$$= \frac{-\mu_0 R^3\sigma\omega\sin\psi\hat{\mathbf{y}}}{2}\int_{-1}^{1} \frac{u}{\sqrt{r^2+R^2-2rRu}}\, du$$

$$\int_{-1}^{+1}\frac{u}{\sqrt{R^2+r^2-2Rru}}\,du = -\frac{(R^2+r^2+Rru)}{3R^2r^2}\sqrt{R^2+r^2-2Rru}\,\Bigg|_{-1}^{+1}$$
$$= -\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\right].$$

$$\mathbf{A}(\mathbf{r}) = \frac{-\mu_0 R^3\sigma\omega\sin\psi\hat{\mathbf{y}}}{2}\left(-\frac{(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)}{3R^2r^2}\right)$$

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{2}(\boldsymbol{\omega}\times\mathbf{r}) & \text{inside} \\[2ex] \dfrac{\mu_0 R^4\sigma}{2r^3}(\boldsymbol{\omega}\times\mathbf{r}) & \text{outside} \end{cases}$$

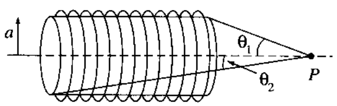
Reverting to the "natural" coordinate, we have

$$\mathbf{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\boldsymbol{\phi}}, & (r \le R), \\[2ex] \dfrac{\mu_0 R^4\omega\sigma}{3}\dfrac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}, & (r \ge R). \end{cases}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{2}{3}\mu_0\sigma R\omega\hat{\mathbf{z}} = \frac{2}{3}\mu_0\sigma R\boldsymbol{\omega}.$$

Surprisingly, the field inside the spherical shell is uniform.

Example 5.12 Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.



Sol: A cute method that does the job.

$$\int \mathbf{B}\cdot d\mathbf{a} = \Phi = \int (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$$

where Φ is the flux of $\mathbf{B}$ through the loop in question.

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \quad\Rightarrow\quad \oint \mathbf{A}\cdot d\mathbf{l} = \Phi$$

Using a circular "amperian loop" at a radius **inside** the solenoid.

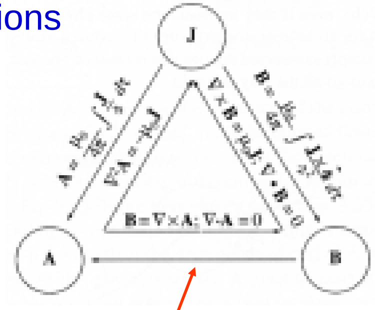
$$\oint \mathbf{A}\cdot d\mathbf{l} = A2\pi s = \int \mathbf{B}\cdot d\mathbf{a} = \mu_0 nI(\pi s^2) \;\Rightarrow A = \frac{\mu_0 nI}{2} s\hat{\phi} \quad \text{for } s<R$$

Using a circular "amperian loop" at a radius s **outside** the solenoid.

$$\oint \mathbf{A}\cdot d\mathbf{l} = A2\pi s = \int \mathbf{B}\cdot d\mathbf{a} = \mu_0 nI(\pi R^2) \;\Rightarrow A = \frac{\mu_0 nIR^2}{2s}\hat{\phi} \quad \text{for } s\ge R$$

5.4.2 Summary; Magnetostatic Boundary Conditions

We have derived *five* formulas interrelating three fundamental quantities: $\mathbf{J}$, $\mathbf{A}$ and $\mathbf{B}$.



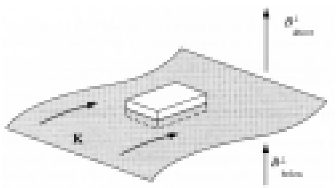
Comments:

- There is one “missing link” in the diagram.
- These three variables, $\mathbf{J}$, $\mathbf{A}$, and $\mathbf{B}$, are all vectors. It is relatively difficult to deal with.

Magnetostatic Boundary Conditions: **Normal**

The magnetic field is not continuous at a surface with surface density $\mathbf{K}$.

What is the physical picture?



Consider a wafer-thin pillbox. Gauss’s law states that

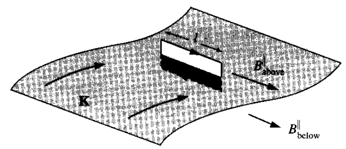
$$\oint_S \mathbf{B}\cdot d\mathbf{a} = 0$$

The sides of the pillbox contribute nothing to the flux, in the limit as the thickness ε goes to zero.

$$(B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}})A = 0 \;\Rightarrow\; B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

Magnetostatic Boundary Conditions: **Tangential**

The tangential component of $\mathbf{B}$ is discontinuous.



Consider a thin rectangular loop. The curl of the Ampere’s law states that $\oint_P \mathbf{B}\cdot d\ell = \mu_0 I_{\text{enc}}$

The ends gives nothing (as ε→0), and the sides give

$$(B^{//}_{\text{above}} - B^{//}_{\text{below}})\ell = \mu_0 K\ell \;\Rightarrow\; B^{//}_{\text{above}} - B^{//}_{\text{below}} = \mu_0 K$$

In short, $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 \mathbf{K}\times\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ points "upward."

How about the vector potential $\mathbf{A}$?

Boundary Conditions in Terms of Vector Potential

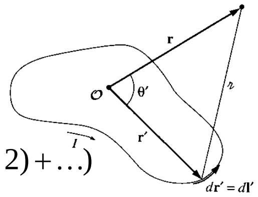
Like the scalar potential in electrostatics, the vector potential is continuous any boundary:

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$$

$$\nabla \cdot \mathbf{A} = 0 \;\Rightarrow\; A^{\perp}_{\text{above}} = A^{\perp}_{\text{below}}$$

$$\nabla \times \mathbf{A} = \mathbf{B} \Rightarrow \oint \mathbf{A}\cdot d\mathbf{l} = \int \mathbf{B}\cdot d\mathbf{a} = \Phi \;\Rightarrow\; A^{//}_{\text{above}} = A^{//}_{\text{below}}$$

5.4.3 Multipole Expansion of the Vector Potential



$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{(r^2 + r'^2 - 2rr'\cos\theta')}}$$
$$= \frac{1}{r}\left(1 + \left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\left((3\cos^2\theta' - 1)/2\right) + \ldots\right)$$
$$= \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} P_{\ell}(\cos\theta')$$

The vector potential of a current loop

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\oint \frac{1}{|\mathbf{r} - \mathbf{r}'|}d\mathbf{l}' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\theta')d\mathbf{l}'$$

$$= \frac{\mu_0 I}{4\pi}\left[\underbrace{\frac{1}{r}\oint d\mathbf{l}'}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\oint r'\cos\theta' d\mathbf{l}'}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\oint r'^2 P_2(\cos\theta')d\mathbf{l}'}_{\text{quadrupole}} + \cdots\right]$$

Multipole Expansion

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\left[\cancel{\frac{1}{r}\oint d\mathbf{l}'} + \frac{1}{r^2}\oint r'\cos\theta' d\mathbf{l}' + \frac{1}{r^3}\oint r'^2 P_2(\cos\theta')d\mathbf{l}' + \cdots\right]$$

magnetic **monopole** term is always zero.

$$\mathbf{A}_{\text{dip}} = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\theta' d\mathbf{l}' = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}'$$

$\boxed{\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}' = -\hat{\mathbf{r}}\times\int d\mathbf{a}'}$ (Eq. 1.108, to be shown later)

Then

$$\mathbf{A}_{\text{dip}} = -\frac{\mu_0}{4\pi}\frac{1}{r^2}\hat{\mathbf{r}}\times\left(I\int d\mathbf{a}'\right) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

where $\mathbf{m} = I\int d\mathbf{a}'$ is the **magnetic dipole moment**.

A Special Technique

Part I Recalling Stokes' theorem $\int_S (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_P \mathbf{v}\cdot d\mathbf{l}$

Let $\mathbf{v}=\mathbf{c}T$ $\quad \nabla\times(f\mathbf{A}) = \nabla f\times\mathbf{A} + f(\nabla\times\mathbf{A})$

$$\int_S (\nabla\times\mathbf{c}T)\cdot d\mathbf{a} = \int_S (\nabla T\times\mathbf{c} + T(\nabla\times\mathbf{c}))\cdot d\mathbf{a} = -\mathbf{c}\cdot\int_S \nabla T\times d\mathbf{a}$$

$$\oint_P \mathbf{c}T\cdot d\mathbf{l} = \mathbf{c}\cdot\oint_P T d\mathbf{l} \quad \Rightarrow \quad \int_S \nabla T\times d\mathbf{a} = -\oint_P T d\mathbf{l}$$

Part II $\quad \int_S \nabla' T'\times d\mathbf{a}' = -\oint_P T' d\mathbf{l}',\;$ let $T' = \hat{\mathbf{r}}\cdot\mathbf{r}'$

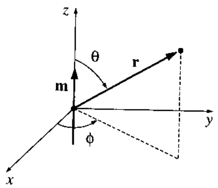
$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

$$\nabla'(\hat{\mathbf{r}}\cdot\mathbf{r}') = \hat{\mathbf{r}}\times\cancel{(\nabla'\times\mathbf{r}')} + \mathbf{r}'\times\cancel{(\nabla'\times\hat{\mathbf{r}})} + (\hat{\mathbf{r}}\cdot\nabla')\mathbf{r}' + \cancel{(\mathbf{r}'\cdot\nabla')\hat{\mathbf{r}}}$$
$$= (\hat{\mathbf{r}}\cdot\nabla')\mathbf{r}' = \hat{\mathbf{r}}$$

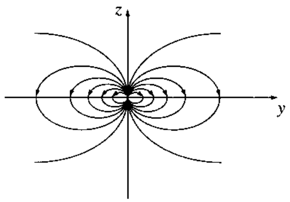
$$\int_S \hat{\mathbf{r}}\times d\mathbf{a}' = -\oint_P (\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}' = \hat{\mathbf{r}}\times\int d\mathbf{a}' \quad \#$$

The Magnetic Field of a Dipole

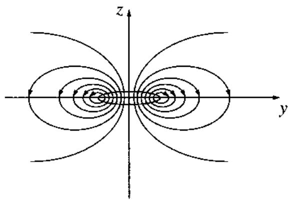
$$\mathbf{A}_{\text{dip}} = \frac{\mu_0}{4\pi}\frac{m\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$



$$\mathbf{B}_{\text{dip}} = \nabla\times\mathbf{A} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$



(a) Field of a "pure" dipole



(a) Field of a "physical" dipole

Homework #10

Problems: 15, 16, 24, 46, 58