Chapter 4 Electric Fields in Matter
4.1 Polarization: 4.1.1 Dielectrics

Most everyday objects belong to one of two large classes: **conductors** and **insulators** (or **dielectrics**)

Conductors : Substances contains an "unlimited" supply of charges that are free to move about through the material.

Dielectrics : all charges are attached to specific atoms or molecules. All they can do is move a bit within the atom or molecule.

Dielectrics

Dielectrics : Microscopic displacements are not as dramatics as the wholesale rearrangement of charge in conductor, but their **cumulative effects** account for the characteristic behavior of dielectric materials.

There are actually two principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule: **stretching** and **rotating**.

4.1.2 Induced Dipoles

What happens to a neutral atom when it is placed in an electric field $\mathbf{E}$?

Although the atom as a while is electrically neutral, there is a *positively* charged core (the nucleus) and a *negatively* charged electron cloud surrounding it.

Thus, the nucleus is pushed in the direction of the field, and the electron the opposite way.

The electric fields *pull* the electron cloud and the nuclear *apart*, their mutual attraction drawing them together - reach balance, leaving the atom polarized.

4.1.2 Induced Dipoles

The atom or molecule now has a tiny dipole moment $\mathbf{p}$, which points in the same direction as $\mathbf{E}$ and is proportional to the field.

$$\mathbf{p} = \alpha\,\mathbf{E}, \qquad \alpha = \text{atomic polarizability}$$

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.76	0.396	24.1	1.64	43.4	59.6

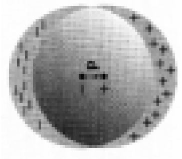
Table 4.1 Atomic Polarizabilities ($\alpha/4\pi\epsilon_0$, in units of 10^{-30} m^3). *Source: Handbook of Chemistry and Physics*, 78th ed. (Boca Raton: CRC Press, Inc., 1997).

Example 4.1 A primitive model for an atom consists of a point nuclear ($+q$) surrounded by a uniformly charged spherical cloud ($-q$) of radius a. Calculate the atomic polarizability of such an atom.

Sol. The actual displacements involved are extremely small. It is reason to assume that the electron cloud retains its spherical shape.

The equilibrium occurs when the nucleus is displaced a distance d from the center of the sphere.

The external field pushing the nucleus to the right exactly balances the internal field pulling it to the left. How?



See Problem 2.18

$$E_e = \frac{1}{4\pi\varepsilon_0}\frac{qd}{a^3} \qquad p = qd = (4\pi\varepsilon_0 a^3)E = \alpha E$$

$$\alpha = 4\pi\varepsilon_0 a^3 = 3\varepsilon_0 v \quad \text{the atomic polarizability}$$

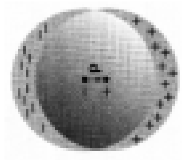
Sol.

The electric field inside a uniform charged sphere of radius a

$$\mathbf{E}_e(r) = \frac{1}{4\pi r^2}\frac{\frac{4}{3}\pi\rho r^3}{\varepsilon_0}\hat{\mathbf{r}} = \frac{1}{3\varepsilon_0}\rho r\hat{\mathbf{r}}$$

$$\therefore\ \mathbf{E}_e(a) = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^3}r\hat{\mathbf{r}}, \quad \text{where } q = \tfrac{4}{3}\pi\rho a^3$$

The electric field produces by two uniform charged spheres separated by $\mathbf{d}$



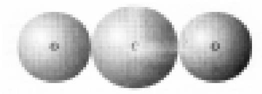
$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{q+}(\mathbf{r}_+) + \mathbf{E}_{q-}(\mathbf{r}_-) = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^3}(\mathbf{r}_+ - \mathbf{r}_-)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{a^3}\left(\left(\mathbf{r} - \frac{1}{2}\mathbf{d}\right) - \left(\mathbf{r} + \frac{1}{2}\mathbf{d}\right)\right) = -\frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{d}}{a^3}$$

$$= -\frac{1}{4\pi\varepsilon_0 a^3}\mathbf{p} \qquad \therefore\ \alpha = 4\pi\varepsilon_0 a^3$$

Polarizability of Molecules

For molecules the situation is not quite so simple, because frequently they polarize more readily in some directions than others.
For instance, carbon dioxide CO_2



When the field is at some angle to the axis, you must resolve it into parallel and perpendicular components, and multiply each by the pertinent polarizability:

$$\mathbf{p} = \alpha_{\perp}\mathbf{E}_{\perp} + \alpha_{/\!/}\mathbf{E}_{/\!/}$$

In this case the induced dipple moment may not even be in the same direction as $\mathbf{E}$.

Polarizability Tensor

CO_2 is relatively simple, as molecules go, since the atoms at least arrange themselves in a straight line.

For a complete asymmetrical molecule, a more general linear relation between $\mathbf{E}$ and $\mathbf{p}$.

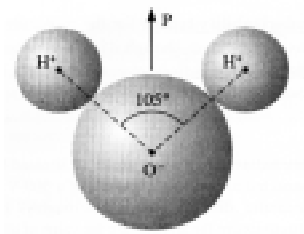
$$p_x = \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z$$
$$p_y = \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z$$
$$p_z = \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z$$

The set of nine constants α_{ij} constitute the polarizability tensor for the molecular.

It is always possible to choose "principal" axies such that the off-diagonal terms vanish, learning just three nonzero polarizabilities.

4.1.3. Alignment of Polar Molecules

The neutral atom has no dipole moment to start with---**p** was induced by the applied field **E**. However, some molecules have *built-in*, *permanent* dipole moment.

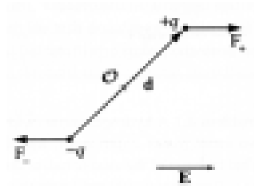


The dipole moment of water is usually large : $6.1*10^{-30}$ C*m, which accounts for its effectiveness as solvent.

What happens when polar molecules are placed in an electric field ? Rotating

Torque for a Permanent Dipole in Uniform Field

In a uniform field, the force on the positive end, $\mathbf{F} = q\mathbf{E}$, exactly cancels the force on the negative end. However, there will be a torque:



$$\mathbf{N} = (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-)$$
$$= [(\mathbf{d}/2) \times (q\mathbf{E}) + (-\mathbf{d}/2) \times (-q\mathbf{E})]$$
$$= q\mathbf{d} \times \mathbf{E}$$

This dipole $\mathbf{p} = q\mathbf{d}$ in a uniform field experiences a torque $\mathbf{N} = \mathbf{p} \times \mathbf{E}$

$\mathbf{N}$ is in such a direction as to line **p** up parallel to **E**, a polar molecule that is free to rotate will swing around until it points in the direction of the applied field.

Net Force due to Field Nonuniformity

If the field is nonuniform, so that $\mathbf{F}_+$ does not exactly balance $\mathbf{F}_-$; There will be a net force on the dipole.

Of course, **E** must change rather abruptly for there to be significant in the space of one molecule, so this is not ordinarily a major consideration in discussing the behavior of dielectrics.

The formula for the force on a dipole in a uniform field is of some interest

@ different position

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ - \mathbf{E}_-) = q(\Delta\mathbf{E}) = q((\mathbf{d}\cdot\nabla)\mathbf{E})$$
$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$$

4.1.4. Polarization

What happens to a piece of dielectric material when it is placed in an electric field?

•Neutral atoms : Inducing tiny dipole moment, pointing in the same direction as the field (**stretching**)

•Polar molecules : experiencing a torque, tending to line it up along the field direction (**rotating**).

Results : A lot of little dipoles points along the direction of the field---the material becomes polarized.

A convenient measure of this of this effect is

$\mathbf{P} \equiv$ dipole moment per unit volume, which is called the polarization.

Prob.4.2 According to quantum mechanics, the electron cloud for a hydrogen atom in ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}$$

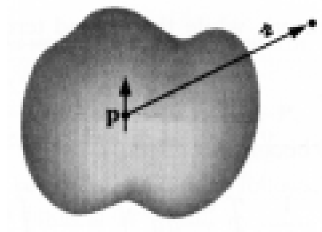
Where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom. [Hint: First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assume $r << a$.

Sol. For a more sophisticated approach, see W. A. Bowers, Am. J. Phys. **54**, 347 (1986).

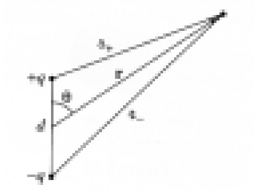
4.2 The Field of a Polarized Object
4.2.1 Bound Charges

Suppose we have a piece of polarized material with polarization $\mathbf{P}$. What is the field produced by this object? (It is easier to work with potential)

For a single dipole $\mathbf{p}$, the potential is $V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\hat{\mathrm{r}} \cdot \mathbf{p}}{\mathrm{r}^2}$ where r is the vector form the dipole to the point at which we are evaluating the potential.



Example 3.10 A electric dipole consists of two equal and opposite charges separated by a distance d. Find the approximate potential at points far from the dipole.



Sol:

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\left|\mathbf{r} - \frac{d}{2}\hat{\mathbf{z}}\right|} - \frac{1}{\left|\mathbf{r} + \frac{d}{2}\hat{\mathbf{z}}\right|}\right) = \frac{q}{4\pi\varepsilon_0 r}\left((1-\varepsilon)^{-1/2} - (1+\varepsilon)^{-1/2}\right)$$

$$\text{where } \varepsilon = \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta'\right) \cong \frac{d}{r}\cos\theta \quad \left(\text{if } \frac{r'}{r} << 1, \text{ so } \theta' \cong \theta\right)$$

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r}\left((1-\varepsilon)^{-1/2} - (1+\varepsilon)^{-1/2}\right) = \frac{q}{4\pi\varepsilon_0 r}\left(\frac{d}{r}\cos\theta\right) = \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{r^2} = \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}$$

where $\mathbf{P} = qd \uparrow$ pointing form negative charge to the positive charge.

4.2.1 Bound Charges

For an infinitesimal dipole moment $d\mathbf{p} = \mathbf{P}d\tau$, the total potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{v} \frac{\hat{\mathrm{r}} \cdot \mathrm{d}\mathbf{p}}{\mathrm{r}^2} = \frac{1}{4\pi\varepsilon_0}\int_{v} \frac{\hat{\mathrm{r}} \cdot \mathbf{P}(\mathbf{r}')}{\mathrm{r}^2}\mathrm{d}\tau'$$

Note that $\nabla'\left(\frac{1}{\mathrm{r}}\right) = \frac{\hat{\mathrm{r}}}{\mathrm{r}^2}$ with respect to the source coordinate.

$$V = \frac{1}{4\pi\varepsilon_0}\int_{v} \underbrace{\mathbf{P} \cdot \nabla'\left(\frac{1}{\mathrm{r}}\right)}_{\text{commutative}} \mathrm{d}\tau'$$

Bound Charges

Integrating by parts, using product rule, gives

$$\boxed{\nabla\cdot\left(\frac{\mathbf{A}}{g}\right)=\frac{g(\nabla\cdot\mathbf{A})-\mathbf{A}\cdot\nabla g}{g^{2}}}$$

$$V=\frac{1}{4\pi\varepsilon_0}\left[\int_v \nabla'\cdot\left(\frac{\mathbf{P}}{\text{r}}\right)d\tau' - \int_v \frac{1}{\text{r}}(\nabla'\cdot\mathbf{P})\,d\tau'\right]$$

$$=\frac{1}{4\pi\varepsilon_0}\oint_S \frac{\mathbf{P}}{\text{r}}\cdot d\mathbf{a}' + \frac{1}{4\pi\varepsilon_0}\int_v \frac{1}{\text{r}}(-\nabla'\cdot\mathbf{P})\,d\tau'$$

$\sigma_b = \mathbf{P}\cdot\hat{n}$ surface charge

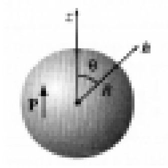
$\rho_b = -\nabla'\cdot\mathbf{P}$ volume charge

Bound Surface and Volume Charges

$$\begin{cases}\sigma_b=\mathbf{P}\cdot\hat{n}\\ \rho_b=-\nabla'\cdot\mathbf{P}\end{cases}\qquad V=\frac{1}{4\pi\varepsilon_0}\oint_S\frac{\sigma_b}{\text{r}}da' + \frac{1}{4\pi\varepsilon_0}\int_v\frac{\rho_b}{\text{r}}d\tau'$$

This means that the potential of a polarized object is the same as that produced by a surface charge plus a volume charge density.

Ex. 4.2 Find the electric field produced by a uniformly polarized sphere of radius R.



$$\begin{cases}\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P\cos\theta'\\ \rho_b = -\nabla'\cdot\mathbf{P} = 0\end{cases}\qquad \boxed{\begin{aligned}da' &= (Rd\theta')(R\sin\theta' d\phi')\\ &= R^2\sin\theta' d\theta' d\phi'\end{aligned}}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^\pi \frac{P\cos\theta'}{\text{r}}2\pi R^2\sin\theta'\cdot d\theta'$$

$$\frac{1}{\text{r}}=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n P_n(\cos\vartheta)\qquad r\ge R$$

$$=\frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\vartheta)\qquad r\le R$$

$\boxed{\vartheta:\text{angle between } \mathbf{r} \text{ and } \mathbf{r}' \Rightarrow \cos\vartheta = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'}$

It will be easier if we let **r** lie on the z axis, so that the angle between them changes from ϑ to θ'

$$\frac{1}{\text{r}}=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n P_n(\cos\theta')\qquad r\ge R$$

$$=\frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\theta')\qquad r\le R$$

$$V(r,0,0)=\frac{1}{4\pi\varepsilon_0}\int_0^\pi \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n P_n(\cos\theta')P\cos\theta' 2\pi R^2\sin\theta' d\theta'\qquad r\ge R$$

$$=\frac{1}{4\pi\varepsilon_0}\int_{-1}^{1} \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n P_n(\cos\theta')P\cos\theta' 2\pi R^2 d\cos\theta'$$

$$=\frac{1}{2\varepsilon_0}\int_{-1}^{1}\frac{PR^3}{r^2}\cos^2\theta' d\cos\theta' = \frac{1}{3\varepsilon_0}\frac{PR^3}{r^2}\qquad \left(\begin{matrix}\text{orthogonality}\\ \text{only } n\text{=1 survive}\end{matrix}\right)$$

$$V(r,0,0)=\frac{1}{4\pi\varepsilon_0}\int_0^\pi \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\theta')P\cos\theta' 2\pi R^2\sin\theta' d\theta'\qquad r\le R$$

$$=\frac{1}{4\pi\varepsilon_0}\int_{-1}^{1}\frac{1}{R}\left(\frac{r}{R}\right)P_1(\cos\theta')\,P\cos\theta' 2\pi R^2 d\cos\theta'$$

$$=\frac{P}{3\varepsilon_0}r\qquad \left(\begin{matrix}\text{orthogonality}\\ \text{only } n\text{=1 survive}\end{matrix}\right)$$

Allow **r** a θ-dependence.

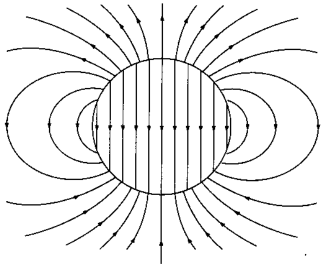
$$V(r,0,0)=\begin{cases}\dfrac{1}{3\varepsilon_0}\dfrac{PR^3}{r^2}\cos\theta & (r>R)\\ \dfrac{P}{3\varepsilon_0}r\cos\theta & (r<R)\end{cases}$$

Electric field of a Uniformly Polarized Sphere

$$V(r) = \frac{P}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta \quad \text{(outside)}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{(\frac{4}{3}\pi P\cdot R^3)}{r^2}\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \quad \text{where } p = \frac{4}{3}\pi R^3 P$$



$$V(r) = \frac{p}{3\varepsilon_0} r\cos\theta = \frac{p}{3\varepsilon_0} z \quad \text{(inside)}$$

Why are the field lines not continuous?

$$\mathbf{E} = -\nabla V = -\frac{P}{3\varepsilon_0}\hat{\mathbf{z}} \quad \text{uniformly}$$

4.2.2 Physical Interpretation of Bound Charges

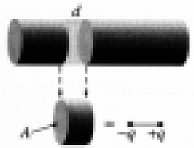
What is the physical meaning of the bound charge?

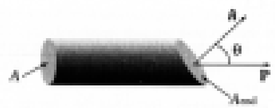
Consider a long string of dipoles.

The net charge at the ends is called the bound charge. The bound charge is no different from any other kind.

Consider a "tube" of dielectric with a given polarization $\mathbf{P}$.

$$\sigma_b = \frac{q}{A} = P \qquad \sigma_b = \frac{q}{A_{end}} = P\cos\theta = \mathbf{P}\cdot\hat{\mathbf{n}}$$





Nonuniform Polarization → The Bound Volume Charge

If the polarization is nonuniform, we get accumulations of bound charge within the material as well as on the surface.

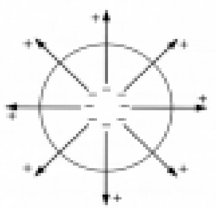
The net bound charge in a given volume is equal and opposite to the amount that has been pushed out through the surface.

$$\int_v \rho_b d\tau = -\oint_S \mathbf{P}\cdot d\mathbf{a} = \int_v (-\nabla\cdot\mathbf{P})\, d\tau$$

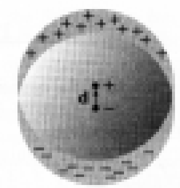
gauss's laws

This is true for any volume bound charge.

$$\rho_b = -\nabla\cdot\mathbf{P}$$



Another Way of Analyzing the Uniformly Polarized Sphere



Example 4.3 Two spheres of charge: a positive sphere and a negative sphere. Without polarization the two are superimposed and cancel completely.

But when the material is uniformly polarized, all the plus charges move slightly upward and all the minus charges move slightly downward.

The two spheres no longer overlap perfectly: at the top there's a "cap" of leftover positive charge and at the bottom a cap of negative charge.

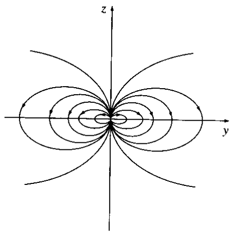
This "leftover" charge is precisely the bound surface charge σ_b.

4.2.3 The Field Inside a Dielectric

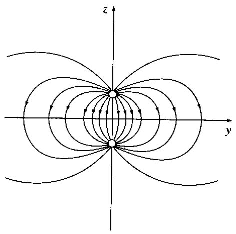
What kind of dipole are we actually dealing with, "pure" dipole or "physical" dipole?

Outside the dielectric there is no real problem, since we are far away from the molecules.

Inside the dielectric, however, we can hardly pretend to be far from all the dipoles.



(a) Field of a "pure" dipole



(a) Field of a "physical" dipole

4.2.3 The Field Inside a dielectric

The electric field inside matter must be very complicated, on the ***microscopic*** level, which would be utterly impossible to calculate, nor would it be of much interest.

The ***macroscopic*** field is defined as the average field over regions large enough to contain many thousand of atoms.

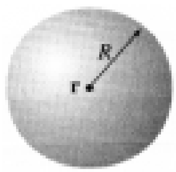
The macroscopic field smoothes over the uninteresting microscopic fluctuation and is *what people mean* when they speak of "the" field inside matter.

The Macroscopic Field

The macroscopic field at $\mathbf{r}$, consists the average field over the sphere due to all charge outside, plus the average due to all charge inside.

$$\mathbf{E} = \mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}}$$

$$V_{\text{out}} = \frac{1}{4\pi\varepsilon_0}\int_{\text{outside}} \frac{\hat{\mathscr{r}} \cdot \mathbf{P}(\mathbf{r}')}{\mathscr{r}^2} d\tau'$$



$$\mathbf{E}_{\text{in}} = -\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{R^3} = -\frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi R^3 \mathbf{P}}{R^3} = -\frac{1}{3\varepsilon_0}\mathbf{P}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{R}\to 0} \frac{\hat{\mathscr{r}} \cdot \mathbf{P}(\mathbf{r}')}{\mathscr{r}^2} d\tau'$$

where the integral runs over the entire volume of the dielectric.

Homework #7

Problems: 2, 9, 10, 16, 31

4.3 The Electric Displacement
4.3.1 Gauss's Law in the Presence of Dielectric

The effect of polarization is to produce accumulation of bound charge, $\rho_b = -\nabla \cdot \mathbf{P}$ within the dielectric and $\sigma_b = \mathbf{P} \cdot \hat{n}$ on the surface.

Now we are going to treat the field caused by both bound charge and free charge. $\rho = \rho_f + \rho_b$

$$= \rho_f - \nabla \cdot \mathbf{P} = \varepsilon_0 \nabla \cdot \mathbf{E}$$

where $\mathbf{E}$ is now the total field, not just that portion generated by polarization . $\varepsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} = \rho_f$

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

Let $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ the electric displacement

Gauss's law reads $\nabla \cdot \mathbf{D} = \rho_f$

Gauss's Law in the Presence of Dielectric

$$\nabla \cdot \mathbf{D} = \rho_f \Rightarrow \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f\,\text{enc}} \leftarrow$$ The total free charge enclosed in the volume

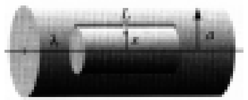
In a typical problem , we know ρ_f , but not ρ_b. So this equation allows us to deal with the information at hand.

What is the contribution of the bound surface charge?

The bound surface charge σ_b can be considered as ρ_b varies rapidly but smoothly within the "skin".

So Gauss's law can be applied elsewhere .

Example 4.4 A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a. Find the electric displacement .



Sol : Drawing a cylindrical Gaussian surface, of radius s and length L, and applying the new Gauss's law , we find

$$\text{Inside } D(2\pi sL) = \lambda L \quad \Rightarrow \quad \mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}} \quad \therefore \mathbf{E} = \frac{\lambda}{2\pi s \varepsilon_r \varepsilon_0}\hat{\mathbf{s}}$$

$$\text{Outside } D(2\pi sL) = \lambda L \quad \Rightarrow \quad \mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}} \quad \therefore \mathbf{E} = \frac{\lambda}{2\pi s \varepsilon_0}\hat{\mathbf{s}}$$

4.3.2 A Deceptive Parallel

"To solve problems involving dielectrics, you just forget all about the bound charge — calculate the field as you ordinarily would, only call the answer $\mathbf{D}$ instead of $\mathbf{E}$"

⇑ This conclusion is false.

For the divergence along is insufficient to determine a vector field; you need to know the curl as well.

$$\nabla \times \mathbf{D} = \varepsilon_0 (\nabla \times \mathbf{E}) + \nabla \times \mathbf{P} = \nabla \times \mathbf{P} \leftarrow \text{not always zero}$$

Since the curl of $\mathbf{D}$ is not always zero, $\mathbf{D}$ cannot be expressed as the gradient of a scalar.

Advice : If the problem exhibits *spherical*, *cylindrical*, or *plane* symmetry, then you can get $\mathbf{D}$ directly from the generalized Gauss's law.

4.3.3 Boundary Conditions

The electrostatic boundary condition in terms of $\mathbf{E}$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$E^{//}_{above} - E^{//}_{below} = 0 \qquad \nabla \times \mathbf{E} = 0$$

The electrostatic boundary condition in terms of $\mathbf{D}$

$$D^{\perp}_{abovt} - D^{\perp}_{below} = \sigma_f \qquad \nabla \cdot \mathbf{D} = \rho_f$$

$$D^{//}_{above} - D^{//}_{below} = P^{//}_{above} - P^{//}_{below} \qquad \nabla \times \mathbf{D} = \nabla \times \mathbf{P}$$

4.4 Linear Dielectric
4.4.1 Susceptibility and Permittivity

For many substances, the polarization is proportional to the field, provided $\mathbf{E}$ is not too strong.

$$P = \varepsilon_0 \chi_e E$$

χ_e :the electric susceptibility of the medium

dimensionless

materials that obey above equation is called linear dielectrics .

The total field $\mathbf{E}$ may be due in part to free charges and in part to the polarization itself .

Permittivity and Dielectric Constant

We cannot compute $\mathbf{P}$ directly from this equation:

the external field will polarize the material

this polarization will produce its own field and contributes to the total field.

$$\mathbf{E}_0 \rightarrow \mathbf{P}_0$$

$$\mathbf{P}_0 \rightarrow \mathbf{E}_0 + \Delta \mathbf{E}'_P$$

The new total field will polarize the material.

$$\mathbf{E}_0 + \Delta \mathbf{E}'_P \rightarrow \mathbf{P}_0 + \Delta \mathbf{P}'_0$$

Will this series converge? Depends.

Linear Media & Dielectric Constant

In linear media ,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E}$$

where, $\varepsilon = \varepsilon_0 (1 + \chi)$ $\qquad \varepsilon_r = \dfrac{\varepsilon}{\varepsilon_0} = 1 + \chi_e$

Permittivity of the material

Relative permittivity or dielectric constant

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7
Neon	1.00013	Salt	5.9
Hydrogen	1.00025	Silicon	11.8
Argon	1.00052	Methanol	33.0
Air (dry)	1.00054	Water	80.1
Nitrogen	1.00055	Ice (-30° C)	99
Water vapor (100° C)	1.00587	$KTaNbO_3$ (0° C)	34,000

Example 4.5 A metal shpere of radius a carries a charge Q. It is surrounded, out to radius b , by linear dielectric material of permittivity ε. Find the potential at the center (relative to infinity).

Sol: Use the generalized Gauss's law

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}} \qquad \text{for all points} \quad r > a$$

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} & \text{for} \quad a < r < b \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} & \text{for} \quad r > b \end{cases}$$

The metal sphere is equalpotential

$$V = -\int_\infty^a \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \frac{Q}{4\pi\varepsilon_0 r^2}dr - \int_b^a \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} + \frac{1}{\varepsilon_r a} - \frac{1}{\varepsilon_r b}\right)$$

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} = \frac{\varepsilon_0\chi_e Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi r^2}\left(\frac{\chi_e}{1+\chi_e}\right)\hat{\mathbf{r}}$$

Cont': Bound Charges in the Dielectric

volume bound charge $\rho_b = -\nabla\cdot\mathbf{P} = 0$

surface bound charge

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \dfrac{\varepsilon_0\chi_e Q}{4\pi\varepsilon b^2} & \text{at the outer surface} \\ \dfrac{-\varepsilon_0\chi_e Q}{4\pi\varepsilon a^2} & \text{at the inner surface} \end{cases}$$

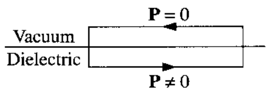
Note that $\hat{\mathbf{n}}$ always points outward with respect to the dielectric , which is $+\hat{\mathbf{r}}$ at *b* but $-\hat{\mathbf{r}}$ at *a* .

The surface bound charge at inner surface is negative. It is this layer of negative charge that reduces the field, within the dielectric by a factor of ε_r .

In this respect a dielectric is rather like an imperfect conductor.

Stokes' Theorem for the Polarization

In general, linear dielectrics cannot escape the defect that $\nabla\times\mathbf{P} \neq 0$



However, if the space is entirely filled with a homogenous linear dielectric, then this objection is void.

$$\nabla\cdot\mathbf{D} = \rho_f$$

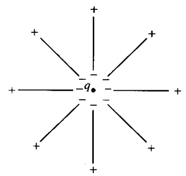
$$\nabla\times\mathbf{D} = 0$$

$$\mathbf{E} = \frac{1}{\varepsilon}\mathbf{D} = \frac{1}{\varepsilon_r}\mathbf{E}_{vac}$$

Remark : When all the space is filled with a homogenous linear dielectric, the field everywhere is simply reduced by a factor of one over the dielectric constant .

Shielding Effect & Susceptibility Tensor

The polarization of the medium partially "shields" the charge, by surrounding it with bound charge of the opposite sign.



For some material, it is generally easier to polarize in some directions than in others .

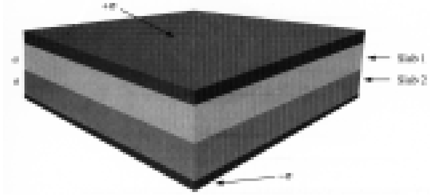
$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \qquad \text{linear dielectric}$$

$$\begin{aligned} P_x &= \epsilon_0(\chi_{e_{xx}}E_x + \chi_{e_{xy}}E_y + \chi_{e_{xz}}E_z) \\ P_y &= \epsilon_0(\chi_{e_{yx}}E_x + \chi_{e_{yy}}E_y + \chi_{e_{yz}}E_z) \\ P_z &= \epsilon_0(\chi_{e_{zx}}E_x + \chi_{e_{zy}}E_y + \chi_{e_{zz}}E_z) \end{aligned}$$

general case
the susceptibility tensor

Prob. 4.18 The space between the planes of a parallel-plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness *a*, so the total distance between the plates is 2*a*. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5 the free charge density on the top plate is σ and on the bottom plate $-\sigma$.

(a) Find the electric displacement **D** in each slab.
(b) Find the electric field **E** in each slab.
(c) Find the polarization **P** in each slab.
(d) Find the potential difference between the plates.
(e) Find the location and amount of all bound charge.
(f) Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (d).



4.4.2 Boundary Value Problems with Linear Dielectrics

Relation between bound charge and free charge

$$\rho_b = -\nabla\cdot\mathbf{P} = -\nabla\cdot\left(\varepsilon_0\chi_e\frac{\mathbf{D}}{\varepsilon}\right) = -\frac{\chi_e}{1+\chi_e}\rho_f \leftarrow \text{in a homogenous linear dielectric}$$

shielding effect

The boundary conditions that makes reference only to the free charge .

$$D^{\perp}_{abovt} - D^{\perp}_{below} = \sigma_f \Rightarrow \varepsilon_{above}E^{\perp}_{above} - \varepsilon_{below}E^{\perp}_{below} = \sigma_f$$

$$(\varepsilon_{\text{above}}\nabla V_{\text{above}} - \varepsilon_{\text{below}}\nabla V_{\text{below}}) = -\sigma_{\text{f}}\hat{\mathbf{n}}$$

$$\text{or}\quad (\varepsilon_{\text{above}}\frac{\partial V_{\text{above}}}{\partial n} - \varepsilon_{\text{below}}\frac{\partial V_{\text{below}}}{\partial n}) = -\sigma_{\text{f}}\quad \text{where}\quad \frac{\partial V_{\text{above}}}{\partial n} = \nabla V\cdot\hat{\mathbf{n}}.$$

Homogeneous Linear Dielectric Sphere

Example 4.7 A sphere of homogeneous linear dielectric material is placed in a uniform electric field **E**. Find the resulant electric field.

Sol: Look at Ex. 3.8 an uncharged conducting sphere. In that case the field of the induced charge ***completely*** canceled **E** within the sphere; However, in a dielectric the cancellation is only ***partial***.

The boundary conditions

(i) $V_{\text{in}} = V_{\text{out}}$, at $r = R$,

(ii) $\epsilon\frac{\partial V_{\text{in}}}{\partial r} = \epsilon_0\frac{\partial V_{\text{out}}}{\partial r}$, at $r = R$, ← no free charge at the surface

(iii) $V_{\text{out}} \to -E_0 r\cos\theta$, for $r \gg R$.

$$V(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$$

$$\begin{cases} V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r \le R \\ V_{out}(r,\theta) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} B_\ell r^{-(\ell+1)}P_\ell(\cos\theta) & r \ge R \end{cases}$$

B.C. (iii)

$$\text{B.C. (i)}:\ A_\ell R^\ell P_\ell = -E_0 R\cos\theta + B_\ell R^{-(\ell+1)}P_\ell$$

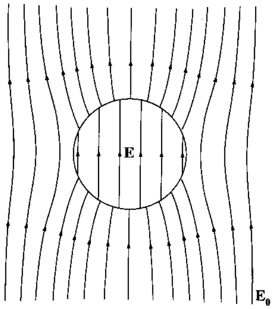
$$\Rightarrow \begin{cases} A_1 R = -E_0 R + B_1 R^{-2} & \ell = 1 \\ A_\ell R^\ell = B_\ell R^{-(\ell+1)} & \ell \ne 1 \end{cases}$$

$$\text{B.C. (ii)}:\ \varepsilon_r \ell A_\ell R^{\ell-1}P_\ell = -E_0\cos\theta - (\ell+1)B_\ell R^{-(\ell+2)}P_\ell$$

$$\Rightarrow \begin{cases} \varepsilon_r A_1 = -E_0 - 2B_1 R^{-3} & \ell = 1 \\ \varepsilon_r \ell A_\ell R^{\ell-1} = -(\ell+1)B_\ell R^{-(\ell+2)} & \ell \ne 1 \end{cases}$$

$$\begin{cases} A_1 R = -E_0 R + B_1 R^{-2} & \ell = 1 \\ A_\ell R^\ell = B_\ell R^{-(\ell+1)} & \ell \neq 1 \end{cases} \qquad \begin{cases} \varepsilon_r A_1 = -E_0 - 2B_1 R^{-3} & \ell = 1 \\ \varepsilon_r \ell A_\ell R^{\ell-1} = -(\ell+1) B_\ell R^{-(\ell+2)} & \ell \neq 1 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = -\dfrac{3E_0}{\varepsilon_r+2}; B_1 = \dfrac{\varepsilon_r - 1}{\varepsilon_r + 2} R^3 & \ell = 1 \\ A_\ell = B_\ell = 0 & \ell \neq 1 \end{cases}$$



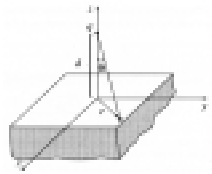
$$\begin{cases} V_{in}(r,\theta) = -\dfrac{3E_0}{\varepsilon_r+2} r\cos\theta \\ V_{out}(r,\theta) = -E_0 r\cos\theta + (\dfrac{\varepsilon_r-1}{\varepsilon_r+2}) R^3 E_0 r^{-2}\cos\theta \end{cases}$$

$$\mathbf{E}_{in} = -\nabla V_{in} = -\frac{3E_0}{\varepsilon_r+2}\hat{\mathbf{z}} \quad \leftarrow \text{ uniform}$$

Partial Image Charge

Example 4.8 Suppose the entire region below the plane z=0 is filled with uniform linear dielectric material of susceptibility χ_e. Calculate the force on a point charge q situated at distant d above the origin.

Sol: The surface bound charge on the xy plane is of opposite sign to q , so the force will be attractive.



z>0 $\quad V = \dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \dfrac{q_b}{\sqrt{x^2+y^2+(z+d)^2}}\right]$

z<o $\quad V = \dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{(q+q_b)}{\sqrt{x^2+y^2+(z-d)^2}}\right] \leftarrow why?$

B.C. $\quad -\varepsilon_0\left(\left.\dfrac{\partial V}{\partial z}\right|_{z=0^+} - \left.\dfrac{\partial V}{\partial z}\right|_{z=0^-}\right) = -\dfrac{1}{2\pi}\left(\dfrac{\chi_e}{\chi_e+2}\right)\dfrac{qd}{(x^2+y^2+d^2)^{3/2}} = \sigma_b$

$$q_b = 2\pi\int_{r=0}^{\infty} -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(r^2+d^2)^{3/2}} r dr = -\frac{1}{2}\int_0^\infty \frac{\chi_e}{\chi_e+2}\frac{qd}{(r^2+d^2)^{3/2}} dr^2 = -\left(\frac{\chi_e}{\chi_e+2}\right) q$$

4.4.3 Energy in Dielectric systems

How to express the energy for a dielectric filled capacitor?

Suppose we bring in the free charge, a bit at a time. As ρ_f is increased by an amount $\Delta\rho_f$, the polarization will charge and with it the bound charge distribution.

The work done on the incremental free charge is :

$$\Delta W = \int (\Delta\rho_f) V d\tau$$

$\nabla\cdot\mathbf{D} = \rho_f \Rightarrow \Delta\rho_f = \nabla\cdot(\Delta\mathbf{D}) \leftarrow$ the resulting change in **D**

$$\Delta W = \int (\nabla\cdot\Delta\mathbf{D}) V d\tau = \int (\overset{0}{\nabla\cdot\Delta\mathbf{D}V} - \nabla V\cdot\Delta\mathbf{D}) d\tau$$

surface integral vanish if we integral will over all of pace.

$$\Delta W = \int \mathbf{E}\cdot\Delta\mathbf{D} d\tau = \frac{1}{2}\int \Delta(\varepsilon\mathbf{E}^2) d\tau \qquad \therefore W = \frac{1}{2}\int (\mathbf{E}\cdot\mathbf{D}) d\tau$$

Which Formula is Correct?

$W = \dfrac{1}{2}\displaystyle\int(\varepsilon_0\mathbf{E}\cdot\mathbf{E})d\tau$ derived in Chap. 2

$W = \dfrac{1}{2}\displaystyle\int(\mathbf{E}\cdot\mathbf{D})d\tau$ derived in Chap. 4

speak to somewhat different question

What do we mean by “the energy of a system“?

It is the work required to assemble the system.

(1) Bring in all the charges (free and bound), one by one, with tweezers, and glue each one down in its proper final position (Chap. 2).

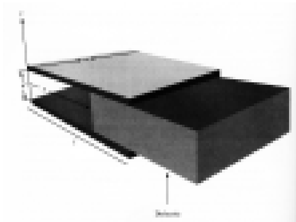
(2) Bring in the free charges, with the unpolarized dielectric in place, one by one, allowing the dielectric to respond as it see fit (Chap. 4).

4.4.4 Forces on Dielectric

The dielectric is attracted into an electric field, just like conductor: the bound charge tends to accumulate near the free charge of the opposite sign.

How to calculate the forces on dielectrics?

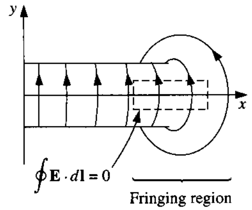
Consider the case of a slab of linear dielectric material, partially inserted between the plates of a parallel-plate capacitor.



If the field is perpendicular to the plates, no force would exert on the dielectric. Is that true?

The Fringing Field Effect

In reality a fringing field around the edges is responsible for the whole effect.
It is this nonuniform fringing field that pulls the dielectric into the capacitor.



Fringing field are difficult to calculate, so we adapt the following ingenious method.

The energy stored in the capacitor is: $W = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$

The electric force on the slab is: $F = -\frac{dW}{dx}$

$$C = C_1 + C_2 = \frac{\varepsilon_0 \omega x}{d} + \frac{\varepsilon_0 \varepsilon_r \omega(\ell - x)}{d} = \frac{\varepsilon_0 \omega}{d}(\varepsilon_r \ell - \chi_e x)$$

Fixed charge

$$F = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx} = -\frac{\varepsilon_0 \chi_e \omega}{2d}V^2$$

$F = -\frac{\varepsilon_0 \chi_e \omega}{2d}V^2 < 0$ indicates that the force is in the negative x direction; the dielectric is pulled into the capacitor.

Fixed voltage $F = +\frac{1}{2}V^2\frac{dC}{dx}$ pushed out. why?

To maintain a constant voltage, the battery must do work.

why? work done by the battery

$$dW = F_{me}dx + VdQ$$

F_{me} : the force I must exert. $(F_{me} = -F)$

$$F = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$$

Homework #8

Problems: 21, 27, 28, 33, 36.