Chapter 3 Special Techniques
3.1 Laplace's Equation: 3.1.1 Introduction

Poisson's equation: $\nabla^2 V = -\frac{1}{\varepsilon_0}\rho(\mathbf{r})$

Very often, we are interested in finding the potential in a region where ρ =0.

There may be plenty of charge elsewhere, but we're confining our attention to places where there is no charge.

Laplace's equation: $\nabla^2 V = 0$

In Cartesian coordinate, $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

3.1.2. Laplace's Equation in 1D

Suppose V depends on only one variable, x.

$$\frac{\partial^2 V}{\partial x^2} = 0 \quad \Rightarrow \quad V(x) = mx + b$$

Two features of this solution:

1. Laplace's equation is a kind of averaging instruction.

$$V(x) = \frac{1}{2}(V(x-a) + V(x+a)) \quad \text{for any } a$$

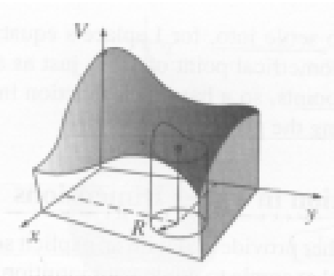
2. Laplace's equation tolerates no local maxima or minima, since the second derivative must be zero.

3.1.3. Laplace's Equation in 2D

Suppose V depends on two variables.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \begin{cases} \text{a partial differential equation (PDE);} \\ \text{not a ordinary differential equation (ODE).} \end{cases}$$

Harmonic functions in two dimensions have the same properties that we noted in one dimension:

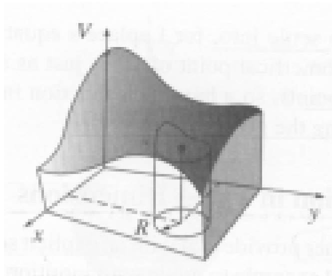


Features of Harmonic Function in 2D

1. The value of V at a point (x, y) is the average of those around the point.

$$V(x, y) = \frac{1}{2\pi R}\oint_{\text{circle}} V d\ell$$

2. V has no local maxima or minima. All extrema occur at the boundaries.



3.1.4. Laplace's Equation in 3D

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0 \quad \text{(partial differential equation (PDE))}$$

In three dimensions we can *neither* provide you with an explicit solution *nor* offer a suggestive physical example to guide your intuition.

Nevertheless, the same two properties remain true.

1. The value of V at a point $\mathbf{r}$ is the average value of V over a spherical ***surface*** of radius R centered at $\mathbf{r}$:

$$V(\mathbf{r})=\frac{1}{4\pi R^2}\oint_{\text{sphere}} V da$$

No Local Maxima or Minima in 3D

2. V has no local maxima or minima; the extreme values must occur at the boundaries.

Ex. For a single point charge q located outside the sphere of radius R as shown in the figure, find the potential at the origin.

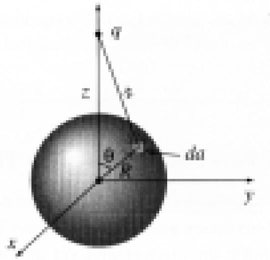
Sol: $V=\frac{1}{4\pi\varepsilon_0}\frac{q}{\text{r}}=\frac{1}{4\pi\varepsilon_0}\frac{q}{(z^2+R^2-2zR\cos\theta)^{1/2}}$

$$\text{so } V_{\text{ave}}(\mathbf{r}=0)=\frac{1}{4\pi R^2}\frac{q}{4\pi\varepsilon_0}\int\frac{R^2\sin\theta d\theta d\phi}{(z^2+R^2-2zR\cos\theta)^{1/2}}$$

$$=\frac{1}{2}\frac{q}{4\pi\varepsilon_0}\int\frac{-d\cos\theta}{(z^2+R^2-2zR\cos\theta)^{1/2}}$$

$$=\frac{1}{2zR}\frac{q}{4\pi\varepsilon_0}(z^2+R^2-2zR\cos\theta)^{1/2}\Big|_0^{\pi}$$

$$=\frac{1}{2zR}\frac{q}{4\pi\varepsilon_0}((z+R)-(z-R))=\frac{q}{4\pi\varepsilon_0 z}$$



3.1.5. Boundary Conditions and Uniqueness Theorems

Laplace's equation does not by itself determine V; a suitable set of boundary conditions must be supplied.

What are appropriate boundary conditions, sufficient to determine the answer and yet not so strong as to generate inconsistencies? It is not easy to see.

For a given set of boundary conditions, is V uniquely determined? Yes, it is. ➔ uniqueness theorem

Boundary Conditions and Uniqueness Theorems

First uniqueness theorem: the solution to Laplace's equation in some volume is uniquely determined if V is specified on the boundary surface.

Proof:

Suppose there were two solutions to

Laplace's equation: $\nabla^2 V_1=0$ and $\nabla^2 V_2=0$

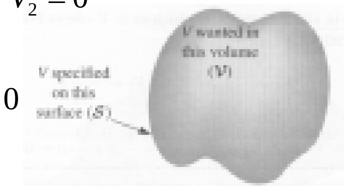
Their difference is: $V_3\equiv V_1-V_2$.

This obays Laplace's equation, $\nabla^2 V_3=0$

Since V_3 is zero on all boundaries and

Laplace's equation suggests that all extrema

occur on the boundary, so V_3. $\Rightarrow V_1=V_2$



Uniqueness Theorems with Charge Inside

$$\nabla^2 V_1 = \frac{\rho}{\varepsilon_0} \quad \text{and} \quad \nabla^2 V_2 = \frac{\rho}{\varepsilon_0}. \text{Let } V_3 \equiv V_1 - V_2 \quad \Rightarrow \quad \nabla^2 V_3 = 0$$

Since V_3 is zero on all boundaries and Laplace's equation suggests that all extrema occur on the boundary, so $V_3 = 0. \Rightarrow V_1 = V_2$

Corollary: The potential in a volume is uniquely determined if (a) the charge density throughout the region, and (b) the value of V on all boundaries, are specified.

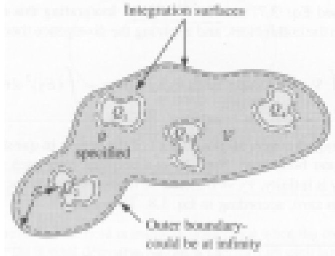
The uniqueness theorem frees your imagination. It doesn't matter how you come by your solution; if (a) it satisfies Laplace's equation and (b) it has the correct value on the boundaries, then it is *right*.

3.1.6. Conductors and the Second Uniqueness Theorems

The *simplest* way to set the boundary conditions for an electrostatic problem is to specify the value of V on all surfaces surrounding the region of interest.

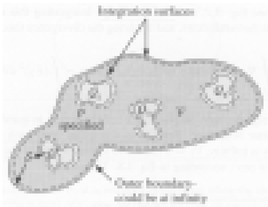
However, in some case we don't know the potential at the boundaries rather the charges on various conducting surfaces. Is the electric field still uniquely determined?

➔ Second uniqueness theorem.



Second Uniqueness Theorems

In a volume surrounded by conductors and containing a specified charge density, the electric field is uniquely determined if the total charge on each conductor is given.



Proof:

Suppose there are two solutions:

$$\nabla \cdot \mathbf{E}_1 = \frac{\rho}{\varepsilon_0} \quad \text{and} \quad \nabla \cdot \mathbf{E}_2 = \frac{\rho}{\varepsilon_0}$$

Both obey Gauss's law in integral form,

$$\oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_1 \cdot da = \frac{1}{\varepsilon_0} Q_i \quad \text{and} \quad \oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_2 \cdot da = \frac{1}{\varepsilon_0} Q_i$$

Likewise, for the outer boundary

$$\oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_1 \cdot da = \frac{1}{\varepsilon_0} Q_{tot} \quad \text{and} \quad \oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_2 \cdot da = \frac{1}{\varepsilon_0} Q_{tot}$$

As before, we examine the difference $\mathbf{E}_3 \equiv \mathbf{E}_1 - \mathbf{E}_2$ which obeys $\nabla \cdot \mathbf{E}_3 = 0$ in the region between the conductors, and $\oint \mathbf{E}_3 \cdot d\mathbf{a} = 0$ over each boundary surface.

Although we don't know how the charge distributes itself over the conducting surface, we do know that each conductor is an equal potential, and hence V_3 is a constant.

Invoking product rule, we find that

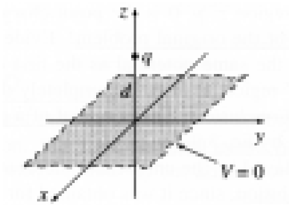
$$\nabla \cdot (V_3 \mathbf{E}_3) = V_3 (\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot \nabla V_3 = -(E_3)^2$$

$$\int_v (\nabla \cdot (V_3 \mathbf{E}_3)) d\tau = \overbrace{\oint_S (V_3 \mathbf{E}_3) \cdot d\mathbf{a}}^{=0} = \int_v -(E_3)^2 d\tau = 0$$

$\therefore E_3 = 0$ everywhere. Consequently, $\mathbf{E}_1 = \mathbf{E}_2$.

3.2 The Method of Images:
3.2.1 The Infinite Grounded Conducting Plane

Suppose a point charge is held a distance d above an infinite grounded conducting plane. What is the potential in the region above the plane?

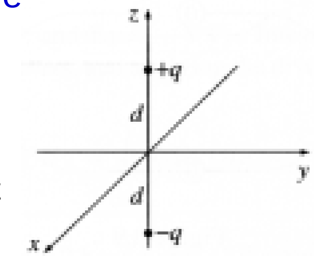


The boundary conditions of this case are:

1. $V = 0$ when z = 0 (since the conducting plane is grounded).
2. $V \to 0$ far from the charge.

The Image Charge

We can easily find a solution which satisfies the boundary conditions as in the figure.



The uniqueness theory guarantees that this case is got to be the right answer.

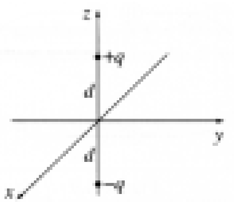
The potential can then be written down as

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Can we use this potential to find out the electric field, surface charge distribution, and the force? Yes.

3.2.2 Induced Surface Charge

It is straightforward to compute the surface charge σ induced on the conductor.



$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n} = -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$$

$$= \frac{-1}{4\pi}\frac{-1}{2}\left[\frac{2(z-d)q}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{2(z+d)q}{(x^2+y^2+(z+d)^2)^{3/2}}\right]_{z=0}$$

$$= \frac{-1}{4\pi}\frac{-1}{2}\frac{-4qd}{(x^2+y^2+d^2)^{3/2}} = \frac{-1}{2\pi}\frac{qd}{(x^2+y^2+d^2)^{3/2}}$$

Total Induced Charge

The total induced charge is (use the polar coordinate)

$$\sigma = \frac{-1}{2\pi}\frac{qd}{(x^2+y^2+d^2)^{3/2}} = \frac{-1}{2\pi}\frac{qd}{(r^2+d^2)^{3/2}}$$

$$Q = \int \sigma da = \int_0^\infty \int_0^{2\pi} \frac{-1}{2\pi}\frac{qd}{(r^2+d^2)^{3/2}} r dr d\phi$$

$$= \int_0^\infty \frac{-qd}{2(r^2+d^2)^{3/2}} dr^2 = \left.\frac{qd}{(r^2+d^2)^{1/2}}\right|_0^\infty = -q$$

3.2.3 Force and Energy

The charge q is attracted toward the plane, because of the negative induced charge.

The force and the energy of this system can be analogous to the case of two point charges.

$$\mathbf{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d^2}\hat{\mathbf{z}}; \quad W = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$$

Unlike the two point charges system, there is no field in the conductor. Handle must be care.

Work and Energy

Consider the work required to bring q in from infinity.

$$W = \int_\infty^d F dz = \int_\infty^d \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4z^2} dz = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}$$

which is half of that of the two point charge system.

This is because the conducting plane is *grounded*.

If the plane is not grounded, what would happen?

3.2.4 The Grounded Spherical Conducting Shell

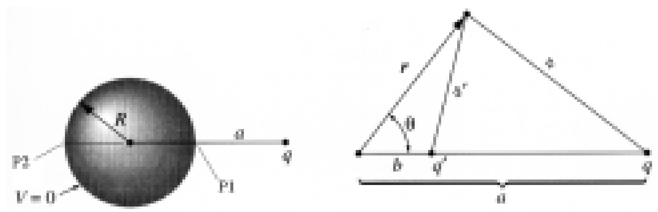
Any stationary charge distribution near a *grounded* conducting plane can be treated in the same way, by introducing its mirror image---**method of images**.

The image charges have *opposite* sign; this is what guarantees that the *plane* will be at potential zero.

Can this method be applied to a curved surface? Yes.

Here is an examples. A point charge is situated in front of a grounded conducting sphere.

Example 3.2 A point charge is situated a distance a from the center of a grounded conducting sphere of radius R. Find the potential outside the sphere.



Sol : Assume the image charge q' is placed at a distance b from the center of the sphere.

It is equipotential on the surface of a grounded sphere.

Using two boundary conditions at P_1 and P_2.

$$\left.\begin{aligned}\text{At } P_1: \frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{R-b}+\frac{q}{a-R}\right)=0\\ \text{At } P_2: \frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{R+b}+\frac{q}{a+R}\right)=0\end{aligned}\right\}\text{two equations and two unknowns } (q' \text{ and } b)$$

$$b=\frac{R^2}{a},\quad q'=-\frac{R}{a}q$$

The force of attraction between charge and the sphere is

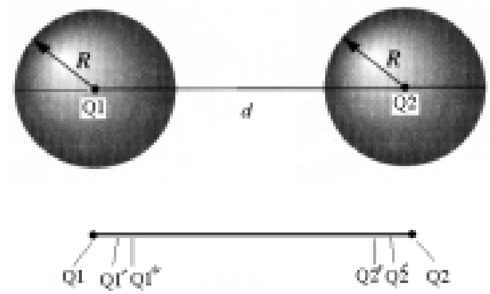
$$F=\frac{1}{4\pi\varepsilon_0}\frac{qq'}{(a-b)^2}=\frac{-1}{4\pi\varepsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}$$

If the sphere is connected to a fixed potential, can this method still be applied? Yes.

Just imagine another image charge situated at the center of the sphere, which provides a constant potential at the surface.

Ex. Two equal conducting spheres with radius R, each carries a total charge Q and $-Q$ at a distance d from each other. Find the electric field outside the conducting spheres.

Sol:



Assume the charges are located at the perspective centers. Using the image charge method, calculate the first level induced charges. Then, calculated the second level induced charges, and so on. The series should converges rather fast.

3.3 Separation of Variables

We shall attack Laplace's equation directly, using the method of separation of variables, which is the physicist's favorite tool for solving partial differential equations.

Applicability: The method is applicable in the circumstances where the potential (V) or the charge density (σ) is specified on the boundaries of some region, and we are asked to the potential in the interior (where ρ =0).

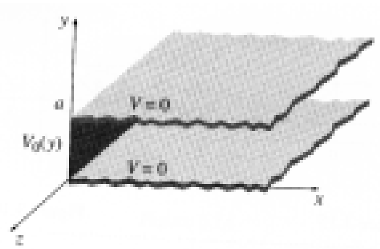
Laplace's equation: $\nabla^2 V = 0$

Basic strategy: look for solutions that are products of functions, each of which depends on only one of the coordinates.

$$V(x,y,z)=X(x)Y(y)Z(z)$$

3.3.1 Cartesian Coordinates

Example 3.3 Two infinite grounded metal plates lies parallel to the xz plane, one at y=0, and the other at $y=a$. The left end, at x=0, is closed off with an infinite strip insulated from the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot".



Boundary Condition

The configuration is independent of z, so Laplace's equation reduces to two dimensions.

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

The potential inside is subject to the boundary conditions.

$$\begin{aligned}
&\text{(i)}\, V=0 \text{ when } y=0,\\
&\text{(ii)}\, V=0 \text{ when } y=a,\\
&\text{(iii)}\, V=V_0(y) \text{ when } x=0,\\
&\text{(iv)}\, V\rightarrow 0 \text{ as } x\rightarrow\infty.
\end{aligned}$$

Separation of Variables

The first step is to look for solutions in the form of products:

$$V(x,y)=X(x)Y(y)$$

Substituting into Laplace's equation, we obtain

$$(Y\frac{\partial^2 X}{\partial x^2}+X\frac{\partial^2 Y}{\partial y^2}=0)\ \times\ \frac{1}{XY}\quad\Rightarrow\quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=0$$

The first term depends only on x and the second only on y. The sum of these two functions is zero, which implies these two functions must both be constant.

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=C_0 \quad\text{and}\quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=-C_0$$

A Simple Solution

Let C_0 equal k^2, for reasons that will appear in a moment.

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=k^2 \quad\Rightarrow\quad X(x)=\overbrace{Ae^{kx}}^{=0\text{ (iv)}}+Be^{-kx}$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=-k^2 \quad\Rightarrow\quad Y(y)=C\sin ky+\overbrace{D\cos ky}^{=0\text{ (i)}}$$

$$V(x,y)=(Ae^{kx}+Be^{-kx})(C\sin ky+D\cos ky)$$

The boundary condition (iv) require A equal zero, and (i) demands that D equal zero.

Meanwhile (ii) yields $\sin ka=0$, from which it follows that

$$k=\frac{n\pi}{a},\quad n=1,2,3,\ldots$$

Why not n=0?

A Complete Solution in Fourier Series

Now we have an infinite set of solutions.

$$V(x,y)=\sum_{n=1}^{\infty}C_n e^{-n\pi x/a}\sin(n\pi y/a)$$

Can we use the remaining boundary condition (iii) to determine the coefficients C_n? Yes.

$$V(0,y)=\sum_{n=1}^{\infty}C_n\sin(n\pi y/a)=V_0(y)$$

This is a Fourier sine series. Virtually any function $V_0(y)$--- can be expanded in such a series. 這麼神奇!

We can use the so-called "Fourier's trick" to find out the coefficients C_n.

The Fourier Trick

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)dy = \int_0^a V_0(y)\sin(n'\pi y/a)dy$$

The integral on the left is

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)dy$$

$$= \frac{1}{2}\int_0^a (\cos((n-n')\frac{\pi y}{a}) - \cos((n+n')\frac{\pi y}{a})dy = \begin{cases} 0, & \text{if } n' \neq n \\ \frac{a}{2}, & \text{if } n' = n \end{cases}$$

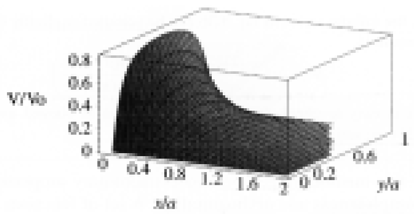
$$C_{n'} = \frac{2}{a}\int_0^a V_0(y)\sin(n'\pi y/a)dy$$

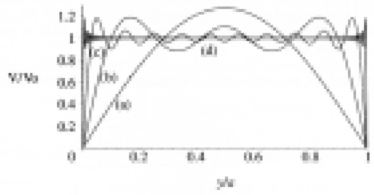
A Concrete Example

For a constant potential V_0

$$C_n = \frac{2V_0}{a}\int_0^a \sin(n'\pi y/a)dy = \frac{2V_0}{n\pi}(1-\cos n\pi) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4V_0}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

So $\quad V(x,y) = \dfrac{4V_0}{\pi}\displaystyle\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a)$





Completeness and Orthogonality

The success of this method hinges on two extraordinary properties, i.e. **completeness** and **orthogonality**.

Completeness: If any other function $f(y)$ can be expressed as a linear combination of a complete function set $f_n(y)$:

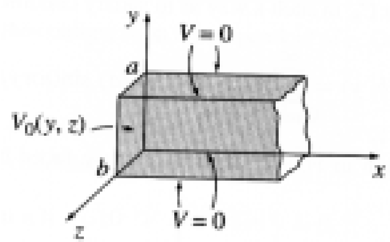
$$f(y) = \sum_{n=1}^{\infty} C_n f_n(y)$$

Orthogonality: If the integral of the product of any two different members of the set is zero:

$$\int_0^a f_n(y) f_{n'}(y)dy = 0 \quad \text{for } n' \neq n$$

Rectangular Metal Pipe

Example 3.5 An infinitely long rectangular metal pipe (side a and b) is grounded, but one end, at $z=0$, is maintained at a specified potential $V_0(y,z)$, as shown in the figure. Find the potential inside the pipe.



Boundary Condition

This is a genuinely three-dimensional problem,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

The potential inside is subject to the boundary conditions.

$$\begin{aligned}
&\text{(i)}\, V=0 \text{ when } y=0,\\
&\text{(ii)}\, V=0 \text{ when } y=a,\\
&\text{(iii)}\, V=0 \text{ when } \mathrm{z}=0,\\
&\text{(iv)}\, V=0 \text{ when } \mathrm{z}=b,\\
&\text{(v)}\, V=V_0(y,z) \text{ when } x=0,\\
&\text{(vi)}\, V\to 0 \text{ as } x\to\infty.
\end{aligned}$$

Separation of Variables

The first step is to look for solutions in the form of products:

$$V(x,y,z)=X(x)Y(y)Z(z)$$

Substituting into Laplace's equation, we obtain

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=0$$

It follows that

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=(k^2+\ell^2),\ \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=-k^2,\ \frac{1}{Z}\frac{\partial^2 Z}{\partial y^2}=-\ell^2$$

How do we know? Any other possibility?

A Simple Solution

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=(k^2+\ell^2)\ \Rightarrow\ X(x)=\overset{=0\ \text{(vi)}}{A}e^{\sqrt{k^2+\ell^2}\,x}+Be^{-\sqrt{k^2+\ell^2}\,x}$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=-k^2\ \Rightarrow\ Y(y)=C\sin ky+\overset{=0\ \text{(i)}}{D}\cos ky$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial y^2}=-\ell^2\ \Rightarrow\ Z(z)=E\sin \ell z+\overset{=0\ \text{(iii)}}{F}\cos \ell z$$

Meanwhile (ii) and (iv) yields $\sin ka=0$ and $\sin \ell b=0$, from which it follows that

$$k=\frac{n\pi}{a},\quad n=1,2,3,\dots\qquad \ell=\frac{m\pi}{b},\quad m=1,2,3,\dots$$

A Complete Solution in Fourier Series

The solution is

$$V(x,y,z)=BCEe^{-\pi\sqrt{(\frac{n}{a})^2+(\frac{m}{b})^2}\,x}\sin(n\pi y/a)\sin(m\pi z/b),$$

where n and m are unspecified integers.

Completeness: the solution can be written as

$$V(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}e^{-\pi\sqrt{(\frac{n}{a})^2+(\frac{m}{b})^2}\,x}\sin(n\pi y/a)\sin(m\pi z/b)$$

Use the boundary condition (v) and the **orthogonality** to find out the coefficients $C_{n,m}$.

$$V(0,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}\sin(n\pi y/a)\sin(m\pi z/b)=V_0(y,z)$$

The Fourier Trick & Constant Voltage Solution

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)dy\int_0^b \sin(m\pi z/b)\sin(m'\pi z/b)dz$$

$$=\int_0^a\int_0^b V_0(y,z)\sin(n'\pi y/a)\sin(m'\pi z/a)dydz$$

$$C_{n,m}=\frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin(n\pi y/a)\sin(m\pi z/b)dydz$$

If the end of the tube is a conductor at constant potential V_0

$$C_{n,m}=\frac{4V_0}{ab}\frac{2a}{n\pi}\frac{2b}{m\pi}=\frac{16V_0}{nm\pi^2} \quad \text{if } n \text{ and } m \text{ are odd.}$$

$$=0 \quad \text{if } n \text{ or } m \text{ are even.}$$

$$V(x,y,z)=\frac{16V_0}{\pi^2}\sum_{n,m=1,3,5,\ldots}^{\infty}\frac{1}{nm}e^{-\pi\sqrt{(\frac{n}{a})^2+(\frac{m}{b})^2}x}\sin(n\pi y/a)\sin(m\pi z/b)$$

Homework #5

Problems: 10, 12, 15, 47

3.3.2 Spherical Coordinates

For round objects spherical coordinates are more suitable.

In the spherical system, Laplace's equation reads

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

We will first treat the problem with **azimuthal symmetry**, so that the potential is independent of ϕ.

$$\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r})+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta})=0$$

Separation of Variables

The first step is to look for solutions in the form of products:

$$V(r,\theta)=R(r)\Theta(\theta)$$

Substituting into spherical Laplace's equation, we obtain

$$\frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r})+\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Theta}{\partial\theta})=0$$

The first term depends only on r and the second only on θ. The sum of these two functions is zero, which implies these two functions must both be constant.

$$\frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r})=\ell(\ell+1), \qquad \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Theta}{\partial\theta})=-\ell(\ell+1)$$

Again, how do we know? Any other possibility?

Simplest Case: A Metal Sphere

Example: A metal sphere of radius R, maintains a specified potential V_0. Find the potential outside the sphere.

Sol: The potential is independent of θ and ϕ.

The Laplace's equation is: $\frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) = 0$

$$R = \frac{A}{r} + B \Rightarrow \frac{\partial R}{\partial r} = -\frac{A}{r^2}$$

$$r^2\frac{\partial R}{\partial r} = -A \Rightarrow \frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) = 0$$

$$R(r = R_0) = A\frac{1}{R_0} + B = V_0 \quad \therefore B = 0 \ (\lim_{r\to\infty} V = 0)$$

$$\underline{\underline{V(r) = V_0\frac{R_0}{r}}} \ \#$$

A Simple Solution & Legendre Polynomials

The general solutions for R and Θ are

$$\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) = \ell(\ell+1)R \quad \Rightarrow \quad R = Ar^{\ell} + B\frac{1}{r^{\ell+1}}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Theta}{\partial\theta}) = -\ell(\ell+1)\Theta$$ The solutions are not simple.

The solutions are **Legendre polynomials** in the variable cosθ.

$$\Theta(\theta) = P_{\ell}(\cos\theta)$$

The polynomials is most conveniently defined by the **Rodrigues formula** (generating function):

$$P_{\ell}(x) = \frac{1}{2^{\ell}\ell!}(\frac{d}{dx})^{\ell}(x^2-1)^{\ell}$$

Rodrigues Formula

Prove: $P_{\ell}(x) = \frac{1}{2^{\ell}\ell!}(\frac{d}{dx})^{\ell}(x^2-1)^{\ell}$, $x=\cos\theta$

where $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial P_{\ell}(\cos\theta)}{\partial\theta}) = -\ell(\ell+1)P_{\ell}(\cos\theta)$

Sol:

Let $v = (x^2-1)^{\ell}$

$$v' = 2\ell x(x^2-1)^{\ell-1} \times (x^2\text{-}1)$$

$$\Rightarrow (1-x^2)v' + 2\ell x v = 0$$

$$(1-x^2)v'' - 2xv' + 2\ell x v' + 2\ell v = 0$$

$$(1-x^2)v'' + 2(\ell-1)xv' + 2\ell v = 0$$

$$(1-x^2)v''' + 2(\ell-2)xv'' + 2(2\ell-1)v' = 0$$

$\cdots \quad (1-x^2)v^{(k+2)} + 2(\ell-k-1)xv^{(k+1)} + (k+1)(2\ell-k)v^{(k)} = 0$

Let $k = \ell$ and $u = v^{(\ell)} = \frac{d^{\ell}(x^2-1)}{dx^{\ell}} = P_{\ell}(\cos\theta)(2^{\ell}\ell!)$

$$\therefore (1-x^2)u'' - 2xu' + \ell(\ell+1)u = 0$$

$$\Rightarrow (1-\cos^2\theta)\frac{d^2P_{\ell}(\cos\theta)}{dx^2} - 2x\frac{dP_{\ell}(\cos\theta)}{dx} + \ell(\ell+1)P_{\ell}(\cos\theta) = 0$$

$$\left\{\begin{aligned} \frac{dP_{\ell}(\cos\theta)}{dx} &= \frac{dP_{\ell}(\cos\theta)}{d\theta}\frac{d\theta}{dx} = -\frac{1}{\sin\theta}\frac{dP_{\ell}(\cos\theta)}{d\theta} \\ \frac{d^2P_{\ell}(\cos\theta)}{dx^2} &= \frac{d}{d\theta}(-\frac{1}{\sin\theta}\frac{dP_{\ell}(\cos\theta)}{d\theta})(-\frac{1}{\sin\theta}) \\ &= \frac{1}{\sin^2\theta}\frac{d^2P_{\ell}(\cos\theta)}{d\theta^2} - \frac{\cos\theta}{\sin^3\theta}\frac{dP_{\ell}(\cos\theta)}{d\theta} \end{aligned}\right.$$

$$(1-\cos^2\theta)\left[\frac{1}{\sin^2\theta}\frac{d^2P_\ell(\cos\theta)}{d\theta^2}-\frac{\cos\theta}{\sin^3\theta}\frac{dP_\ell(\cos\theta)}{d\theta}\right]$$
$$-2\cos\theta\left[-\frac{1}{\sin\theta}\frac{dP_\ell(\cos\theta)}{d\theta}\right]+\ell(\ell+1)P_\ell(\cos\theta)$$
$$=\frac{d^2P_\ell(\cos\theta)}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{dP_\ell(\cos\theta)}{d\theta}+\ell(\ell+1)P_\ell(\cos\theta)$$
$$=\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_\ell(\cos\theta)}{d\theta}\right)+\ell(\ell+1)P_\ell(\cos\theta)=0$$

$$\therefore\quad \underline{\underline{P_\ell(\cos\theta)=\frac{1}{2^\ell\ell!}\frac{d^\ell(\cos^2\theta-1)}{dx^\ell}}}\ \#$$

Properties of Legendre Polynomials

The first few Legendre polynomials are listed

$$\begin{aligned}P_0(x)&=1\\P_1(x)&=x\\P_2(x)&=(3x^2-1)/2\\P_3(x)&=(5x^3-3x)/2\\P_4(x)&=(35x^4-30x^2+3)/8\\P_5(x)&=(63x^5-70x^3+15x)/8\end{aligned}$$

$P_\ell(x)$: an ℓth - order polynomial in x

Completeness: The Legendre polynomials constitute a complete set of function, on the interval $-1\le x\le 1$.

Orthogonality: The polynomials are orthogonal functions:

$$\int_{-1}^{1}P_\ell(x)P_{\ell'}(x)dx=\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta d\theta$$
$$=\begin{cases}0 & \text{if } \ell'\ne\ell\\ \dfrac{2}{2\ell+1}, & \text{if } \ell'=\ell\end{cases}$$

A Complete Solution in Legendre Polynomials

The Rodrigues formula generates only one solution. What and where are other solutions?

These "other solutions" blow up at $\theta=0$ and/or $\theta=\pi$, are therefore unacceptable on physical grounds.

$$V(r,\theta)=\left(Ar^\ell+B\frac{1}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

The general solutions is the linear combination of separable solutions.

$$V(r,\theta)=\sum_{\ell=0}^{\infty}\left(Ar^\ell+B\frac{1}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Example 3.6 The potential $V_0(\theta)=V_0\sin^2(\theta/2)$ is specified on the surface of a hollow sphere, of radius R. Find the potential inside the sphere.

Sol: In this case $B_\iota=0$ for all ι---otherwise the potential would blow up at the origin. Thus,

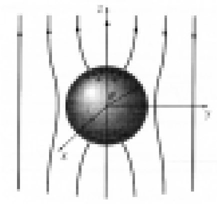
$$V(r,\theta)=\sum_{\ell=0}^{\infty}A_\ell r^\ell P_\ell(\cos\theta)\ \rightarrow\ V(R,\theta)=\sum_{\ell=0}^{\infty}A_\ell R^\ell P_\ell(\cos\theta)$$
$$A_\ell=\frac{2\ell+1}{2}\frac{1}{R^\ell}\int_0^\pi V(R,\theta)P_\ell(\cos\theta)\sin\theta d\theta$$
$$=\frac{2\ell+1}{2}\frac{1}{R^\ell}\int_0^\pi V_0\sin^2\left(\frac{\theta}{2}\right)P_\ell(\cos\theta)\sin\theta d\theta$$
$$=\frac{2\ell+1}{2}\frac{1}{R^\ell}\int_0^\pi \frac{V_0}{2}(1-\cos\theta)P_\ell(\cos\theta)\sin\theta d\theta$$
$$=\frac{2\ell+1}{2}\frac{1}{R^\ell}\int_0^\pi \frac{V_0}{2}(P_0(\cos\theta)-P_1(\cos\theta))P_\ell(\cos\theta)\sin\theta d\theta$$

$$A_\ell = \frac{2\ell+1}{2}\frac{1}{R^\ell}\int_0^\pi \frac{V_0}{2}(P_0(\cos\theta) - P_1(\cos\theta))P_\ell(\cos\theta)\sin\theta\, d\theta$$

$$\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)dx = \begin{cases} 0 & \text{if } \ell' \neq \ell \\ \dfrac{2}{2\ell+1}, & \text{if } \ell' = \ell \end{cases}$$

$$\begin{aligned} A_0 &= \frac{V_0}{2} \\ A_1 &= -\frac{V_0}{2R} \end{aligned} \quad \Rightarrow \quad V(r,\theta) = \frac{V_0}{2}\left[1 - \frac{r}{R}\cos\theta\right]$$

Example 3.8 An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. Find the potential in the region outside the sphere.



Sol: The sphere is an *equipotential*---we may as well set it to zero.

The potential is *azimuthal symmetric* and by symmetry the entire xy plane is at potential zero.

In addition, the potential is not zero at large z.

Boundary conditions are:

$$\text{(i)}\ V = 0 \text{ when } r = R,$$

$$\text{(ii)}\ V \to -E_0 r\cos\theta \ \text{ for } r >> R.$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$$

$$\text{B.C. (i):}\ \ V(R,\theta) = \sum_{\ell=0}^{\infty}(A_\ell R^\ell + B_\ell R^{-(\ell+1)})P_\ell(\cos\theta) = 0$$

$$\Rightarrow B_\ell = A_\ell R^{2\ell+1}$$

$$\text{B.C. (ii):}\ \ V(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell)P_\ell(\cos\theta) = -E_0 r\cos\theta$$

$$\Rightarrow A_1 = -E_0, \quad \text{all other } A_\ell \text{ are zero.}$$

$$\left\{ \begin{aligned} & V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta \\ & \mathbf{E}\big|_{r=R} = -\nabla V = E_0\left(1 + 2\frac{R^3}{R^3}\right)\cos\theta\hat{\mathbf{r}} = 3E_0\cos\theta\hat{\mathbf{r}} \\ & \sigma(\theta) = \varepsilon_0(3E_0\cos\theta\hat{\mathbf{r}})\hat{\mathbf{r}} = 3\varepsilon_0 E_0\cos\theta \end{aligned} \right.$$

3.4 Multiple Expansion

3.4.1 Approximate Potential at Large Distance

If you are very far from a localized charge distribution, it "looks" like a point charge, and the potential is---to good approximation—$(1/4\pi\varepsilon_0)Q/r$, where Q is the total charge. But what if Q is zero?

Develop a systematic expansion for the potential of an arbitrary localized charge distribution, in powers of $1/r$.

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{|\mathbf{r} - \mathbf{r}'|}\rho(\mathbf{r}')d\tau'$$

Using the law of cosines,

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{(r^2 + (r')^2 - 2rr'\cos\theta')}}$$

Note, for simplicity, $\mathbf{r} = r\hat{\mathbf{z}}$

Large Distance Approximation

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{\sqrt{(r^2+(r')^2-2rr'\cos\theta'}}=\frac{1}{r}(1+\varepsilon)^{-1/2}$$

where $\varepsilon=\frac{r'}{r}(\frac{r'}{r}-2\cos\theta')$ Taylor's expansion

$$\frac{1}{r}(1+\varepsilon)^{-1/2}=\frac{1}{r}(1-\frac{1}{2}\varepsilon+\frac{3}{8}\varepsilon^2-\frac{5}{16}\varepsilon^3+\ldots),\ \text{if } \varepsilon<<1$$

So $$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{r}(1-\frac{1}{2}\frac{r'}{r}(\frac{r'}{r}-2\cos\theta')+\frac{3}{8}(\frac{r'}{r}(\frac{r'}{r}-2\cos\theta'))^2$$

$$-\frac{5}{16}(\frac{r'}{r}(\frac{r'}{r}-2\cos\theta'))^3+\ldots)$$

$$=\frac{1}{r}(1+(\frac{r'}{r})\cos\theta'+(\frac{r'}{r})^2((3\cos^2\theta'-1)/2)+\ldots)$$

Legendre Polynomials & Multiple Expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{r}(1+(\frac{r'}{r})\cos\theta'+(\frac{r'}{r})^2((3\cos^2\theta'-1)/2)+\ldots)$$

$$=\frac{1}{r}\sum_{\ell=0}^{\infty}(\frac{r'}{r})^{\ell}P_{\ell}(\cos\theta')$$

$$V(\mathbf{r})=\int\frac{1}{4\pi\varepsilon_0 r}\sum_{\ell=0}^{\infty}(\frac{r'}{r})^{\ell}P_{\ell}(\cos\theta')\rho(\mathbf{r}')d\tau'$$ This is the desired result.

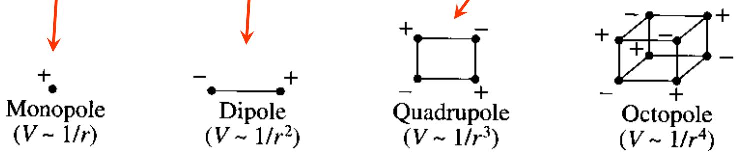
$$=\frac{1}{4\pi\varepsilon_0 r}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell}}\int(r')^{\ell}P_{\ell}(\cos\theta')\rho(\mathbf{r}')d\tau'$$

or more exolicity $$\boxed{V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\left[\begin{array}{l}\frac{1}{r}\int\rho(\mathbf{r}')d\tau'+\frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')d\tau'\\+\frac{1}{r^3}\int(r')^2(\frac{3}{2}\cos^2\theta'-1)\rho(\mathbf{r}')d\tau'+\ldots\end{array}\right]}$$

The multiple expansion of *V* in power of 1/*r*.

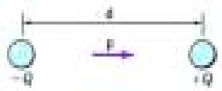
Legendre Polynomials & Multiple Expansion

$$V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}(\underline{\frac{1}{r}\int\rho(\mathbf{r}')d\tau'}+\underline{\frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')d\tau'}+\underline{\frac{1}{r^3}\int(r')^2(\frac{3}{2}\cos^2\theta'-1)\rho(\mathbf{r}')d\tau'}+\ldots)$$



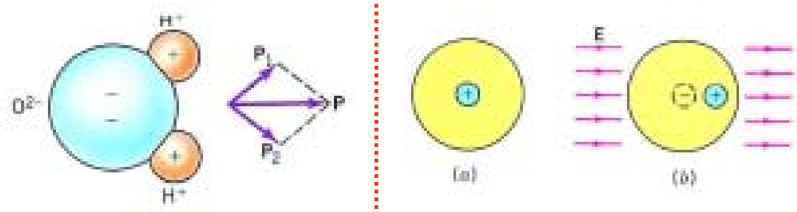
Dipoles

What is dipole? The arrangement of a pair of equal and opposite charges separated by some distance is called an electric dipole.

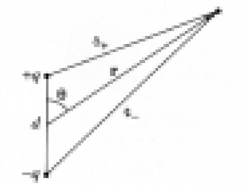


Permanent dipole: such as molecules of HCl, CO, and H_2O.

Induced dipole: An electric field may also induce a charge separation in an atom or a nonpolar molecule.



Example 3.10 A electric dipole consists of two equal and opposite charges separated by a distance d. Find the approximate potential at points far from the dipole.



Sol:

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\mathbf{r} - \frac{d}{2}\hat{\mathbf{z}}|} - \frac{1}{|\mathbf{r} + \frac{d}{2}\hat{\mathbf{z}}|}\right) = \frac{q}{4\pi\varepsilon_0 r}((1-\varepsilon)^{-1/2} - (1+\varepsilon)^{-1/2})$$

$$\text{where } \varepsilon = \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta'\right) \cong \frac{d}{r}\cos\theta \quad \left(\text{if } \frac{r'}{r} << 1, \text{ so } \theta' \cong \theta\right)$$

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r}((1-\varepsilon)^{-1/2} - (1+\varepsilon)^{-1/2})$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left(\frac{d}{r}\cos\theta\right) = \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}$$

The Electric Field of a Dipole

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2} = \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}$$

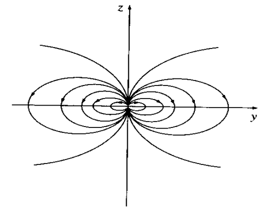
Why?

Just a convention.

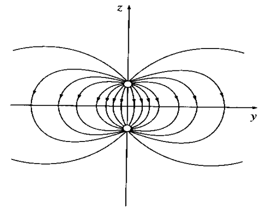
where $\mathbf{P} = qd \uparrow$ pointing form negative charge to the positive charge.

$$\mathbf{E} = -\nabla V(\mathbf{r}) = \frac{p}{4\pi\varepsilon_0}\left(-\frac{2\cos\theta}{r^3}\hat{\mathbf{r}} + \frac{\sin\theta}{r^2}\hat{\boldsymbol{\theta}} + 0\hat{\boldsymbol{\varphi}}\right)$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}(-2\cos\theta\hat{\mathbf{r}} + r\sin\theta\hat{\boldsymbol{\theta}})$$



(a) Field of a "pure" dipole

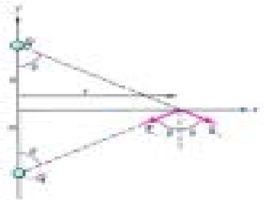


(a) Field of a "physical" dipole

Some *Important* Properties of Dipole

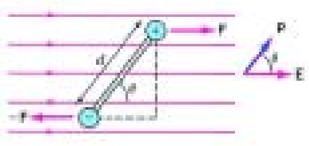
Field due to a dipole:

$$\mathbf{p} = Q\mathbf{d}\ (- \rightarrow +)$$



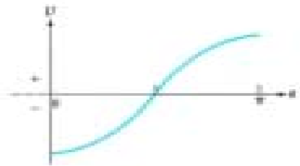
Torque in a uniform field:

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$$



Potential energy:

$$U = -\mathbf{p}\cdot\mathbf{E}$$



Homework #6

Problems: 19, 26, 37, 49